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Robust stability of positive discrete-time interval systems with time-delays

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Abstract. Necessary and sufficient conditions for robust stability of the positive discrete-time interval system with time-delays are established. It is shown that this system is robustly stable if and only if one well defined positive discrete-time system with time-delays is asymptotically stable. The considerations are illustrated by numerical example.

Keywords: robust stability, linear system, positive system, interval system, discrete-time, time-delay.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1,2]. Recent developments in positive systems theory and some new results are given in [3]. The reachability of positive discrete-time systems with time-delays has been considered in [4,5] and the minimal energy control of the same class of positive systems has been studied in [6].

The stability and robust stability problems of linear system with time-delays has been considered in many papers and books, see for example [7–10]. These books are directed to the non-positive systems.

Robust stability problem of continuous and discrete-time system (in general non-positive) was considered in [11,12], for example. Existing stability criteria for interval non-positive discrete-time systems described by the state-space equations have the forms of only sufficient conditions.

In this paper, we extend the main results of [13] to positive discrete-time interval systems with time-delays. It will be shown that interval positive system described by the state-space equations is robustly stable if and only if one well defined positive discrete-time system with time-delays is asymptotically stable.

To the best authors knowledge the above problem for positive interval discrete-time systems with time-delays has been not considered yet.

2. Preliminaries

Let $\Re^{n\times m}$ be the set of $n\times m$ matrices with entries from the field of real numbers and $\Re^n=\Re^{n\times 1}$. The set of $n\times m$ matrices with real non-negative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n=\Re^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider the positive discrete-time linear system with delays described in the state space by the homogeneous equation

$$\boldsymbol{x}_{i+1}=\sum_{k=0}^{h}\boldsymbol{A}_k\boldsymbol{x}_{i-k},\quad i\in Z_+,\tag{1}$$

with the initial conditions

$$\boldsymbol{x}_{-i},\quad i=0,1,...,h,\tag{2}$$

where h is a positive number, $\boldsymbol{x}_i\in\Re^n$ is the state vector and

$$\boldsymbol{A}_k\in\Re_+^{n\times n}\quad(k=0,1,...,h).\tag{3}$$

If (3) holds then for every $\boldsymbol{x}_{-i}\in\Re_+^n$ $(i=0,1,...,h)$ we have $\boldsymbol{x}_i\in\Re_+^n$ for $i\in Z_+$ [4, 13].

The system (1) is asymptotically stable if and only if all roots $z_1,z_2,...,z_n$ of the characteristic equation

$$\det(z\boldsymbol{I}_n-\sum_{k=0}^{h}\boldsymbol{A}_k z^{-k})=0\tag{4a}$$

have moduli less than 1, or equivalently, all roots $z_1,z_2,...,z_{\tilde{n}}$ of the equation

$$\det\left(z^{h+1}\boldsymbol{I}_n-\sum_{k=0}^{h}\boldsymbol{A}_k z^{h-k}\right)=z^{\tilde{n}}+a_{\tilde{n}-1}z^{\tilde{n}-1}+...+a_1z+a_0=0\tag{4b}$$

have moduli less than 1, i.e.

$$|z_k|<1\quad\text{for } k=1,2,...,\tilde{n}=(h+1)n.\tag{5}$$

Defining

$$\vec{\boldsymbol{x}}_i=\begin{bmatrix}\boldsymbol{x}_i\\ \boldsymbol{x}_{i-1}\\ \vdots\\ \boldsymbol{x}_{i-h+1}\\ \boldsymbol{x}_{i-h}\end{bmatrix}\in\Re_+^{\tilde{n}},\quad \vec{\boldsymbol{x}}_0=\begin{bmatrix}\boldsymbol{x}_0\\ \boldsymbol{x}_{-1}\\ \vdots\\ \boldsymbol{x}_{-h+1}\\ \boldsymbol{x}_{-h}\end{bmatrix}\in\Re_+^{\tilde{n}},\tag{6}$$

the equation (1) can be written in the form

$$\vec{\boldsymbol{x}}_{i+1}=\boldsymbol{A}\vec{\boldsymbol{x}}_i,\quad i\in Z_+,\tag{7}$$

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with initial condition $\vec{x}_0$, where $\tilde{n} = (h+1)n$ and

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_{h-1} & \boldsymbol{A}_h \\ \boldsymbol{I}_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \boldsymbol{I}_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times\tilde{n}}. \quad (8)$$

It is well known [1, 2] that the positive system (7) is asymptotically stable if and only if all eigenvalues $z_1, z_2, ..., z_{\tilde{n}}$ of the matrix $\boldsymbol{A}$ (the roots of the equation $\det(z\boldsymbol{I}_{\tilde{n}} - \boldsymbol{A}) = 0$) have moduli less than 1.

It was shown that [13]

$$\det(z\boldsymbol{I}_{\tilde{n}} - \boldsymbol{A}) = \det(z^{h+1}\boldsymbol{I}_n - \sum_{k=0}^{h} \boldsymbol{A}_k z^{h-k}). \quad (9)$$

This means that asymptotic stability of the system (1) (with delays) is equivalent to asymptotic stability of the system (7) (without delays).

THEOREM 1 [13]. The positive system with time-delays (1) is asymptotically stable if and only if the following equivalent conditions hold:

1) all coefficients of the polynomial

$$\det \boldsymbol{M}_h(z) = \det[(z+1)^{h+1}\boldsymbol{I}_n - \sum_{k=0}^{h} \boldsymbol{A}_k (z+1)^{h-k}]$$
$$= z^{\tilde{n}} + \tilde{a}_{\tilde{n}-1} z^{\tilde{n}-1} + ... + \tilde{a}_1 z + \tilde{a}_0 \quad (10)$$

are positive, i.e. $\tilde{a}_i > 0$ for $i = 1, 2, ..., \tilde{n}-1$

2) all leading (principal) minors of the matrix $\bar{\boldsymbol{A}} = \boldsymbol{I}_{\tilde{n}} - \boldsymbol{A}$ are positive.

3. Robust stability

Let us consider a family of positive discrete-time linear systems with delays described by

$$\boldsymbol{x}_{i+1} = \sum_{k=0}^{h} \boldsymbol{A}_k \boldsymbol{x}_{i-k}, \quad \boldsymbol{A}_k \in [\boldsymbol{A}_k^-, \boldsymbol{A}_k^+] \subset \Re_+^{n\times n}$$
$$\text{for } k = 0, 1, ..., h. \quad (11)$$

The positive system (11) is called an interval positive system with time delays.

The interval positive system (11) is called robustly stable if the system (1) is asymptotically stable for all $\boldsymbol{A}_k \in [\boldsymbol{A}_k^-, \boldsymbol{A}_k^+] \subset \Re_+^{n\times n}$ $(k = 0, 1, ..., h)$.

If $\boldsymbol{A}_k \in [\boldsymbol{A}_k^-, \boldsymbol{A}_k^+] \subset \Re_+^{n\times n}$ for all $k = 0, 1, ..., h$ then

$$\boldsymbol{A} \in \boldsymbol{A}_I = [\boldsymbol{A}^-, \boldsymbol{A}^+] \subset \Re_+^{\tilde{n}\times\tilde{n}}, \quad (12)$$

where $\boldsymbol{A}$ is of the form (8) and

$$\boldsymbol{A}^- = \begin{bmatrix} \boldsymbol{A}_0^- & \boldsymbol{A}_1^- & \cdots & \boldsymbol{A}_{h-1}^- & \boldsymbol{A}_h^- \\ \boldsymbol{I}_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \boldsymbol{I}_n & 0 \end{bmatrix},$$
$$\boldsymbol{A}^+ = \begin{bmatrix} \boldsymbol{A}_0^+ & \boldsymbol{A}_1^+ & \cdots & \boldsymbol{A}_{h-1}^+ & \boldsymbol{A}_h^+ \\ \boldsymbol{I}_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \boldsymbol{I}_n & 0 \end{bmatrix}. \quad (13)$$

THEOREM 2. The interval positive time-delays system (11) is robustly stable if and only if the positive system without delays

$$\vec{\boldsymbol{x}}_{i+1} = \boldsymbol{A}^+ \vec{\boldsymbol{x}}_i, \quad i \in Z_+, \quad (14a)$$

is asymptotically stable or, equivalently the positive time-delays system is asymptotically stable

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}_0^+ \boldsymbol{x}_0 + \sum_{k=1}^{h} \boldsymbol{A}_k^+ \boldsymbol{x}_{i-k}, \quad i \in Z_+. \quad (14b)$$

Proof. The proof follows directly from the fact that all eigenvalues of any non-negative matrix $\boldsymbol{A} \in [\boldsymbol{A}^-, \boldsymbol{A}^+]$ have moduli less than 1 if and only if all eigenvalues of $\boldsymbol{A}^+$ have moduli less than 1 [14,15] (see also [11,16]).

From Theorem 2 it follows that robust stability of interval system (11) does not depend on the matrices $\boldsymbol{A}_k^- \in \Re_+^{n\times n}$ $(k = 0, 1, ..., h)$. Therefore may be $\boldsymbol{A}_k^- = 0$, $k = 0, 1, ..., h$. Moreover, if the system (1) is asymptotically stable for fixed $\boldsymbol{A}_k = \boldsymbol{A}_{kf} \in \Re_+^{n\times n}$, $k = 0, 1, ..., h$, then this system is also asymptotically stable for all $\boldsymbol{A}_k \in [0, \boldsymbol{A}_{kf}]$, $k = 0, 1, ..., h$.

The system (14b) is asymptotically stable if and only if all roots $z_1, z_2, ..., z_n$ of the equation

$$\det\left(z^{h+1}\boldsymbol{I}_n - \sum_{k=0}^{h} \boldsymbol{A}_k^+ z^{h-k}\right)$$
$$= z^{\tilde{n}} + \bar{a}_{\tilde{n}-1} z^{\tilde{n}-1} + ... + \bar{a}_1 z + \bar{a}_0 = 0 \quad (15)$$

have moduli less than 1.

From the above and Theorem 1 we have the following theorem.

THEOREM 3. The interval positive time-delays system (11) is robustly stable if and only if the following equivalent conditions hold:

1) all coefficients of the polynomial

$$\det[(z+1)\boldsymbol{I}_{\tilde{n}} - \boldsymbol{A}^+] = \det \boldsymbol{M}_h^+(z)$$
$$= z^{\tilde{n}} + \hat{a}_{\tilde{n}-1} z^{\tilde{n}-1} + ... + \hat{a}_1 z + \hat{a}_0 \quad (16)$$

are positive, i.e. $\hat{a}_i > 0$ for $i = 0, 1, ..., \tilde{n}-1$, where $\tilde{n} = (h+1)n$, has the form given in (13) and

$$\boldsymbol{M}_h^+(z) = (z+1)^{h+1}\boldsymbol{I}_n - \sum_{k=0}^{h} \boldsymbol{A}_k^+ (z+1)^{h-k}. \quad (17)$$

2) all leading (principal) minors of the matrix

$$\bar{\boldsymbol{A}}^{+} = \boldsymbol{I}_{\tilde{n}} - \boldsymbol{A}^{+} = \begin{bmatrix} \boldsymbol{I}_n - \boldsymbol{A}_0^{+} & -\boldsymbol{A}_1^{+} & \cdots & -\boldsymbol{A}_{h-1}^{+} & -\boldsymbol{A}_h^{+} \\ -\boldsymbol{I}_n & \boldsymbol{I}_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{I}_n & 0 \\ 0 & 0 & \cdots & -\boldsymbol{I}_n & \boldsymbol{I}_n \end{bmatrix} \quad (18)$$

are positive.

THEOREM 4. The interval positive time-delays system (11) is not robustly stable if 1) the positive system (without delays)

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}_0^{+}\boldsymbol{x}_i, i \in Z_+, \quad (19)$$

is unstable, or

2) at least one diagonal entry of the matrix $\boldsymbol{A}_0^{+} = [a_{ij}^{0+}]$ is greater than 1, i.e.

$$a_{kk}^{0+} > 1 \quad \text{for some } k \in (1, 2, ..., n). \quad (20)$$

P r o o f . In [13] it was shown that instability of the positive system (without delays)

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}_0\boldsymbol{x}_i, \quad i \in Z_+, \quad (21)$$

always implies the instability of the positive system with time-delays (1). The proof of 1) follows from the fact that asymptotic stability of (21) with $\boldsymbol{A}_0 \in [\boldsymbol{A}_0^{-}, \boldsymbol{A}_0^{+}] \subset \Re_+^{n\times n}$ is equivalent to asymptotic stability of the system (21) with $\boldsymbol{A}_0 = \boldsymbol{A}_0^{+}$.

The positive system (1) is unstable if at least one diagonal entry of the matrix $\boldsymbol{A}_0 = [a_{ij}^0]$ is greater than 1 [13]. Hence, if $\boldsymbol{A}_0 \in [\boldsymbol{A}_0^{-}, \boldsymbol{A}_0^{+}] \subset \Re_+^{n\times n}$ the above condition is satisfied if (20) holds.

4. Example

Consider the positive interval system

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}_0\boldsymbol{x}_i + \boldsymbol{A}_1\boldsymbol{x}_{i-1}, \quad (22)$$

where $\boldsymbol{A}_k \in [\boldsymbol{A}_k^{-}, \boldsymbol{A}_k^{+}] \subset \Re_+^{3\times 3}$ for $k = 0$ and $k = 1$ with

$$\boldsymbol{A}_0^{-} = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \boldsymbol{A}_0^{+} = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.2 & 0 & a \\ 0 & 0.1 & 0 \end{bmatrix}, (23a)$$

$$\boldsymbol{A}_1^{-} = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}, \boldsymbol{A}_1^{+} = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.4 & 0 & 0 \\ 1 & 0 & b \end{bmatrix}. (23b)$$

Find values of the parameters $a \geqslant 0$ and $b \geqslant 0$ for which the system (22), (23) is robustly stable.

In this case the matrix $\bar{\boldsymbol{A}}^{+} = \boldsymbol{I}_6 - \boldsymbol{A}^{+}$ has the form

$$\bar{\boldsymbol{A}}^{+} = \begin{bmatrix} \boldsymbol{I}_3 - \boldsymbol{A}_0^{+} & -\boldsymbol{A}_1^{+} \\ -\boldsymbol{I}_3 & \boldsymbol{I}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.2 & 0 & 0 & -0.2 & 0 \\ -0.2 & 1 & -a & -0.4 & 0 & 0 \\ 0 & -0.1 & 1 & -1 & 0 & -b \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

Computing all leading (principal) minors Δ_i^{+}, $(i = 1, 2, ..., 6)$ of the matrix (24), from condition 2) of Theorem 3 we obtain:

$\Delta_1^{+} = 1 > 0$, $\Delta_2^{+} = 0.96 > 0$,
$\Delta_3^{+} = 0.96 - 0.1a > 0$ hence $a < 9.6$,
$\Delta_4^{+} = -0.3a + 0.88 > 0$ hence $a < 2.9333$,
$\Delta_5^{+} = -0.5a + 0.76 > 0$ hence $a < 1.52$,
$\Delta_6^{+} = \det \bar{\mathbf{A}}^{+} = -0.5a + 0.76 - 0.76b > 0$ hence $b < 1 - 0.6579a$.

From the above it follows that the interval system (22) is robustly stable for $0 \leqslant a < 1.52$, $0 \leqslant b < 1 - 0.6579a$. If $a = 1$, for example, then $0 \leqslant b < 0.3421$.

The same result can be obtained by the use of the condition 1) of Theorem 3.

5. Concluding remarks

It has been shown that the interval positive discrete-time system with time-delays, described by (11), is robustly stable if and only if one well defined positive discrete-time system with time-delays (14b) is asymptotically stable.

Necessary and sufficient conditions and also simple necessary conditions for robust stability checking are given.

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BusÅ‚owicz M., Robust stability of positive discrete-time linear systems with multiple delays with unity rank uncertainty structure or non-negative perturbation matrices, Bull. Pol. Acad.Â BusÅ‚owicz M., Stability of positive singular discrete-time system with unit delay with canonical forms of state matrices, Proc. 12th IEEE int. Conf. on Methods and Models in Automation and Robotics, MiÄ™dzyzdroje 2006, pp. 215-218. Time-delay phenomena are usually appeared in many practical systems, such as AIDS epidemic, chemical engineering systems, hydraulic systems, inferred grinding model, neural network, nuclear reactor, population dynamic model, and rolling mill. Hence stability analysis and. stabilization for discrete switched systems with time delay have been researched in recent years [2-4, 7, 9-10, 12-14].Â In [7], a switching signal design technique is proposed to guarantee the asymptotic stability of discrete switched systems with interval time-varying delay. In [14], the switching signal is identified to guarantee the stability of discrete switched time-delay system. Approach to delay-dependent robust stability and stabilisation of delta operator systems with time-varying delays. Access Full Text. Approach to delay-dependent robust stability and stabilisation of delta operator systems with time-varying delays. Author(s): Hicham El Aiss 1 ; Hafsaa Rachid 1 ; Abdelaziz Hmamed 1 ; Ahmed El Hajjaji 2. DOI: 10.1049/iet-cta.2017.0006.Â 20. Zong, G., Hou, L.: â€˜New delay-dependent stability result and its application to robust performance analysis for discrete-time systems with delayâ€™, IMA J. Math. Control Inf., 2010, 27, pp. 373â€“386. 7). 28. Li, X., Gao, H.: â€˜A new model transformation of discrete-time systems with time-varying delay and its application to stability analysisâ€™, IEEE Trans. Automatic Control, 2011, 56, (9), pp. 2172â€“2178. 8).