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Modified Lévy measures of Hinde-Demétrio processes

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Abstract. The Hinde-Demétrio processes, recently introduced to extend the negative binomial ones, are particular cases of Poisson stopped-sum processes. Probability mass and characteristic functions are not commonly usable. However, they are useful overdispersed models and admit simple variance functions $m+m^p$. We show that, for all $p \geq 2$, the modified Lévy measures ρ of these stochastic processes still belong in the negative binomial family with variance function $m+m^2$, where $x^{-2}[\rho(dx)-\rho(\{0\})\delta_0(dx)]$ are their associated Lévy measures.

Keywords: Compound Poisson process, Laplace transform, Lévy process, variance function.

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§1. Introduction and result

The Hinde-Demétrio process $\mathscr{X}_p = \{X_{p,t}\ ;\, t \geq 0\}$, $p > 1$, is the Poisson stopped-sum (or compound Poisson) process

$$X_{p,t} = \sum_{i=1}^{N_t} U_{p,i} \text{ where } \mathbb{E}(z^{U_{p,i}}) = z \times \frac{{}_2F_1\left(\frac{1}{p-1}, \frac{1}{p-1}; \frac{p}{p-1}; (qz)^{p-1}\right)}{{}_2F_1\left(\frac{1}{p-1}, \frac{1}{p-1}; \frac{p}{p-1}; q^{p-1}\right)}$$

for $0 < q < 1$, $\mathscr{N} = \{N_t\ ;\, t \geq 0\}$ is the standard Poisson process with intensity $\lambda > 0$, independent of $U_{p,i}$, $i = 1, 2, \ldots$ (sizes of jumps), and

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

is the Gaussian hypergeometric function (cf. [2]). Kokonendji *et al.* [4] have shown that the distribution of $X_{p,t}$ belongs to the family of probability measures

$$F_{p,\lambda t} = F(\mu_p^{*\lambda t}) = \left\{\exp(\theta x)\, \mu_p^{*\lambda t}(dx) \,\Big|\, \int_{\mathbb{R}} \exp(\theta x)\mu_p^{*\lambda t}(dx); \theta \in \Theta \subset \mathbb{R}\right\},$$

called natural exponential family (NEF) generated by $\mu_p^{*\lambda t}$ (i.e. λt-th power of convolution of μ_p). Thus, the characteristical variance function of the NEF $F_p = F(\mu_p)$ is simply

$$V_{F_p}(m) = m + m^p, \;\; m \in (0, \infty). \tag{1}$$

As particular cases we have the negative binomial law for $p = 2$ and the strict arcsine law for $p = 3$ (cf. [5]). For more details on NEFs and their variance functions, the reader can consult for instance the Chapter 54 of the book [7].

The Hinde-Demétrio process, as particular case of the Lévy processes (cf. [1, 9, 10]), can be defined in terms of its Lévy measure of type 0 (terms to be precised below) or in terms of its modified Lévy measure ρ such that $x^{-2}[\rho(dx) - \rho(\{0\})\delta_0(dx)]$ is the associated Lévy measure. Using the theorem below (due to authors [6] and that can be extended to multidimensional situation), we see that

$$V_{G_p}(\overline{m}) = \frac{p-1}{p}(\overline{m}-1)(\overline{m}-1+p) \tag{2}$$

on $M_{G_p} = (1,\infty)$, where $G_p = G(\rho_p)$ is the NEF generated by the modified Lévy measure $\rho = \rho(\mu_p)$ of μ_p. Hence, the NEF G_p is of the negative binomial family for all $p > 1$.

Theorem 1. *Let $F = F(\mu)$ be the NEF generated by μ with variance function V_F on M_F. Assume that the modified Lévy measure $\rho = \rho(\mu)$ of μ generates the NEF $G = G(\rho)$. Then, the variance function V_G of G is such that for all $m \in M_F$*

$$V_G \circ V_F'(m) = V_F(m)V_F''(m). \tag{3}$$

Moreover, if $V_F(m) \sim m^p$, $p \geq 2$, as $m \to \infty$ (the Tweedie class at infinity [3]), then $V_G(\overline{m}) \sim \overline{m}^2$ as $\overline{m} = V_F'(m) \to \infty$ (the Morris class at infinity [8]).

Let us precise now some definition about Lévy measure of a Lévy process $\mathscr{X} = \{X_t\,; t \geq 0\} \equiv \mathscr{X}_\mu$ governed by an infinitely divisible law μ, which is a probability measure whose nth roots (in the convolution sense) exist for any positive integer n.

At first, we recall that a Lévy process $\mathscr{X}_\mu$ is a càdlàg stochastic process with stationary and independent increments and $X_0 = 0$. More frequently, the behaviour of a Lévy process $\mathscr{X} = \mathscr{X}_\mu$ is provided by the well-known Lévy-Khintchine characterization: a probability measure μ on $\mathbb{R}$ is infinitely divisible if and only if there exist $\gamma, \sigma \in \mathbb{R}$ and a positive finite measure ν on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}} \min\left(1, x^2\right) \nu(dx) < \infty \tag{4}$$

such that the Fourier transform of μ is of the form

$$\int_{\mathbb{R}} e^{i\theta x}\mu(dx) = \exp\left\{ i\gamma\theta - \frac{\sigma^2\theta^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \left[e^{i\theta x} - 1 - i\theta\tau(x)\right] \nu(dx) \right\}, \tag{5}$$

where τ is some fixed bounded continuous function on $\mathbb{R}$ such that $(\tau(x) - x)/x^2$ is bounded as $x \to 0$ (e.g. $\tau(x) = x/(1+x^2)$ or $\tau(x) = \sin x$). In this case the triplet (γ, σ^2, ν) is unique, and the measure $\nu = \nu(\mu)$ satisfying (4) is called the *Lévy measure* of μ. From (5), the first characteristic γ is connected to the drift of the process $\mathscr{X}_\mu$, whereas σ^2 is the infinitesimal variance of the Brownian motion part of $\mathscr{X}_\mu$, and ν determines the probabilistic character of the jumps of $\mathscr{X}_\mu$. For example, a compound Poisson process is a Lévy process $\mathscr{X}$ with $\gamma = \sigma = 0$ and ν a finite measure. When ν is not a finite measure, $\mathscr{X}$ is not a compound Poisson process; so $\mathscr{X}$ has infinitely many jumps in every finite time interval of strictly positive length. Consequently, one may define infinitely divisible distributions in terms of their Lévy measure. Recall here that the Lévy measure $\nu = \nu(\mu)$ is classified from (4) in three types as follows: If ν is bounded then it is said to be of type 0. If ν is unbounded and $\int_{\mathbb{R}\setminus\{0\}} \min(1,|x|)\,\nu(dx) < \infty$ it is said to be of type 1. It is said to be of type 2 if $\int_{\mathbb{R}\setminus\{0\}} \min(1,|x|)\,\nu(dx)$ diverges.

§2. Concluding remarks

The fact that the modified Lévy measure of any Hinde-Demétrio process still remains in the negative binomial family (2) would be difficult to obtain through the Lévy-Khintchine characterization (5). The compound Poisson process is completely characterized by the knowledge of the (modified) Lévy measure. In particular, the result (2) provides a new characterization of the Hinde-Demétrio processes.

Finally, observing that for $p \in \{2,3,\ldots\}$, the Hinde-Demétrio processes can be used for modelling overdispersed count data; e.g. the number of claims reported to an insurance company during a period of time [4]. One practical question that we can ask is the following. What is meaning (for overdispersed count data) that all Hinde-Demétrio processes admit the same negative binomial family as for their modified Lévy measures?

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5. LÃ©vy Processes from Poisson Point Processes. 5.1. Poisson processes. 5.2. Compound Poisson Processes.Â Since a LÃ©vy process has right-continuous sample paths, for each xed argument âœ“ the function t 7! 't (âœ“) is right-continuous, and in particular, since '0(âœ“) = 1Â Thus, to obtain the integral of any function f against the random measure N (d y , !) one just sums the values of f at the points in P (!). In the case of a compound Poisson process, the relevant intensity measure on R2 is. (5.6). Âµ(d t , d y) = d t d F (y). Thus, in this case the intensity measure assigns mass t to the strip t := {(s , y) : s t }. Since this mass is nite, the number of points N (t) of the corresponding Poisson point process in the strip t is nite.