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Monomial cubature rules since “Stroud”: a compilation — part 2

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1. Introduction

More than 25 years ago, Stroud published his encyclopedic work on multiple numerical integration, *Approximate Calculation of Multiple Integrals* [98]. This book contains a listing of almost all multiple integration or cubature rules for a variety of regions known at that time. About 5 years ago, *Monomial cubature rules since* “*Stroud*”: *a compilation* [104] was published. This paper was concerned with continuing the work of Stroud in one specific area, namely the compilation of all the so-called monomial cubature rules which have appeared since the publication of [98] for most of the regions contained there plus some cubature rules which appeared earlier but were not included in [98] for some reason.

The study of multiple numerical integration has continued during the past years, although we notice a slow-down. Nevertheless, we believe that this is the right time to come out with a small list of errata and addenda. We also use the occasion to present some cubature rules that are of theoretical importance but are not well known.

For background material, a description of the material included and the tables we refer to Cools and Rabinowitz [104]. References given in [104] are not repeated in this paper. So for references from [1] to [101], the reader must look at [104]. The numbering of the new references starts at [102].

Readers with access to Russian literature were referred to the book by Mysovskikh [81]. Indeed, as a rule, we ignored the Russian literature except when it appeared in translation. Meanwhile a

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German translation of this book became available as a report [113]. We include a reference to this report for all cubature rules that were not available in English publications before 1981, i.e, when [81] appeared.

2. Errata and addenda to the tables

If something changed for a particular region, degree and number of points N, we give here *the complete corresponding part of the table*. This is done because this is the easiest way to present some errata and addenda.

The tables have an extra column to emphasize the changes. Newly added references can immediately be recognised by a reference number larger than 101. New cubature rules got an additional "new!" in the last column. Lines that contain corrections, are labelled with a "cor" in the last column.

The only changes with respect to the tables of embedded cubature rules are: (1) the report [22] is now published as a journal article [103] and (2) new rules for the entire space are derived in [106].

For convenience, we briefly repeat the notation for quality of the cubature rule. The first letter gives information about the weights. If all weights are positive, this is denoted by the letter P. If some weights are negative, this is indicated by the letter N. If the weights are not explicitly given, this is indicated by a question mark. The second letter appears only for bounded regions and gives information about the location of the integration points. If all points are inside the region, this is indicated by the letter I unless some of the points lie on the boundary in which case the letter B is used. If some points are outside the region, an O is written. An asterisk is appended to the number of points if this is known to be the theoretically minimal number of points.

2.1. Cubature formulae for two-dimensional regions

2.1.1. Cubature formulae for the square C_2

Degree	N	Quality	References	Remark
3	4*	PI	[98]	
		PI(∞)	[118]	new!
	5	PI	[102]	new!
		NI	[82]	
4	6*	PI	[91, 101]	cor
		PI	[101]	cor
		PO	[65]	
5	7*	PI(2)	[98]	cor
11	24*	PI	[18, 117]	
19	68	NO	[99, 116]	
23	112	NI	[116]	new!

2.1.2. Cubature formulae for the circle S_2

Degree	N	Quality	References	Remark
6	10*	PO	[90, 101]	
		PO	[62]	cor
	11	PO	[101, 113]	
8	16	PI	[101, 113]	
9	19	PI	[74]	
		PI	[51]	
		PI	[113]	new!
	20	PI(3)	[11]	
		PO	[11]	
		PO	[11, 110]	
13	36	PI	[15, 110]	
15	44	PI	[98]	
		PI	[110]	new!
17	60	PO	[110]	new!
19	72	PI	[110]	new!

2.1.3. Cubature formulae for the triangle T_2

Degree	N	Quality	References	Remark
4	6*	PI	[23, 29, 64, 68, 73, 97]	
		?I	[92]	
		PB	[113]	new!
6	10*	PO	[114,108]	new!
		PO	[108]	new!

2.2. Cubature formulae for the plane $E_2^{r^2}$

Degree	N	Quality	References	Remark
9	18*	P	[51]	
		P	[109]	new!

2.3. Cubature formulae for three-dimensional regions

2.3.1. Cubature formulae for the cube C_3

Degree	N	Quality	References	Remark
3	6*	PB	[98]	
		PI	[107]	new!
5	13*	PI	[98]	
		PI	[77]	
		PI	[2]	
		PI(3)	[107]	new!
		PO	[113,107]	new!
7	31	NI	[111]	new!
	38	PI	[111]	new!
8	47	PI	[111]	new!
9	53	NO	[72]	
		NO	[31]	
		NO	[111]	new!
	63	PB	[36]	
		PI	[36]	
		PI	[111]	new!
10	77	NO	[111]	new!
11	89	NO	[31]	
		NI	[36]	
		NO	[111]	new!
	91	PI(2)	[36]	
		PB(2)	[36]	
		PI	[111]	new!
12	127	NO	[111]	new!
	131	NO	[111]	new!
15	205	NI	[116]	new!

2.3.2. Cubature formulae for the sphere S_3

Degree	N	Quality	References	Remark
5	14	PO	[48]	cor

2.3.3. Cubature formulae for the tetrahedron T_3

Degree	N	Quality	References	Remark
4	11	NI	[59, 113]	
11	87	NB	[112]	new!

2.4. Cubature formulae for four-dimensional regions

2.4.1. Cubature formulae for the 4-cube C_4

Ref. [27] does not contain a cubature rule of degree 7 with 73 points. The rules in [27] are only valid for dimensions $n \geqslant 5$. See also Section 2.5.

2.4.2. Cubature formulae for the 4-sphere S_4

Degree	N	Quality	References	Remark
4	16	PB	[113]	new!
5	30	PO(2)	[113]	new!

2.5. Cubature formulae for n-dimensional regions

Region	Degree	N	Quality	References	Remark
C_n $(n \geqslant 5)$	7	$(4n^3 - 6n^2 + 14n + 3)/3$	NI	[27]	cor
S_n $(n = 5, 6)$	5	$(n+1)(n+2)$	PI(2)	[113]	new!
S_n $(n \geqslant 3)$	7	$(4n^3 - 6n^2 + 14n)/3$	PI if $n = 3$ PO if $n = 4, 5$ NI if $n \geqslant 6$	[113]	new!
S_n $(n \geqslant 3)$	7	33 $(n = 3)$; 73 $(n = 4)$ 101 $(n = 5)$; 293 $(n = 6)$; $(4n^3 - 6n^2 + 14n + 3)/3$, $n \geqslant 7$	PI if $n \leqslant 4$ NI if $n \geqslant 5$	[27]	cor
S_n $(n \geqslant 5)$	7	123 $(n = 5)$ $(n^3 + 9n^2 + 14n + 9)/3$, $n \geqslant 6$	NO if $5 \leqslant n \leqslant 12$ NI if $13 \leqslant n \leqslant 40$	[115]	new!
S_n $(n \geqslant 5)$	7	123 $(n = 5)$ $(n^3 + 9n^2 + 14n + 9)/3$, $n \geqslant 6$	NO if $5 \leqslant n \leqslant 7$ NI if $8 \leqslant n \leqslant 40$	[115]	new!
T_n $(4 \leqslant n \leqslant 12)$	4	$(n^2 + 3n + 4)/2$	?I	[113]	new!
T_n $(n \geqslant 4)$	7	$(n^3 + 12n^2 + 29n + 24)/6$	?I	[113,105]	new!

For completeness we mention that the formula for T_n in [105] was copied from Mysovskikh's work.

Embedded cubature formulae

Region	Degrees	N	Quality	References	Remark
$E_n^{r^2}$	1-3-···-51	See ([106], p. 305)	?(2)	[106]	new!

3. Three special cubature rules

As a rule, we ignored the Russian and other non-western literature except when it has appeared in translation. We made one exception to this rule [62] justified by the theoretical importance of the

Table 1
15-point formula of degree 8 for T_2

i	w_i	x_i	y_i
1	0.066940767639916174192	0.51334692063945414949	0.28104124731511039057
2	0.043909556791220782402	0.31325121067172530696	0.63062143431895614010
3	0.029285717640165892159	0.65177530364879570754	0.31347788752373300717
4	0.026530624434780379347	0.065101993458939166328	0.87016510156356306078
5	0.16058343856681218798 10^{-9}	0.34579201116826902882	3.6231682215692616667

cubature rule. We make a second exception here [114] for the same reason. We also include here a table with the points and weights of a minimal cubature rule for T_2 that appears in [17] which is not in the accessible literature.

3.1. A minimal rule of degree 6 for the triangle

For T_2 and degree 6, the minimal number of points is 10. A cubature rule that attains this bound was already obtained in 1983 by Rasputin [114]. It is invariant under coordinate interchange and has 2 points outside the region.

Rasputin knew there were two solutions but he was not interested in the second rule: he knew a priori that one point was outside the region. Both cubature rules were recomputed by [108].

3.2. A minimal rule of degree 6 for the circle

For S_2 and degree 6, the minimal number of points is 10. A cubature rule that attains this bound was already obtained in 1986 by Rasputin [90] and Wissman and Becker [101]. It is invariant under coordinate interchange and has 2 points outside the region.

In 1989, Konyaev and Tolmacheva [62] obtained a cubature rule with no symmetry and also 2 points outside the region.

If one removes the restriction that a cubature rule has to be invariant under coordinate interchange, for S_2, degree 6 and 10 points, one introduces one degree of freedom. Probably both known cubature rules belong to the same continuum of minimal cubature rules. It remains an open question whether this continuum has a cubature rule on it with all points inside the circle or not.

3.3. A minimal rule of degree 8 for the triangle

For T_2 and degree 8, the minimal number of points is 15. A cubature rule that attains this bound was obtained in 1987 by Cools and Haegemans [17]. Since this rule has only appeared in an internal report, we give the points and weights in Table 1.

It is invariant under the "rotations" of the triangle and has 3 points outside the region. It thus has the form

$$Q[f] = \sum_{i=1}^{5} w_i(f(x_i, y_i) + f(y_i, 1 - x_i - y_i) + f(1 - x_i - y_i, x_i)).$$

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