

OPTICAL SOLITONS IN BIREFRINGENT FIBERS WITH FOUR-WAVE
MIXING FOR KERR LAW NONLINEARITY

MICHELLE SAVESCU^{1,2},
A.H. BHRAWY^{3,4}, E.M. HILAL⁵, A.A. ALSHAERY⁵, ANJAN BISWAS^{1,3}

¹Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA

² Department of Mathematics, Kutztown University of Pennsylvania,
15200 Kutztown Road, Kutztown, PA-19530, USA

³Department of Mathematics, Faculty of Science, King Abdulaziz University,
Jeddah 21589, Saudi Arabia

⁴Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt

⁴Department of Chemical and Materials Engineering, Faculty of Engineering,
King Abdulaziz University, Jeddah 21589, Saudi Arabia

⁵Department of Mathematics, Faculty of Science for Girls, King Abdulaziz University,
Jeddah-21589, Saudi Arabia

Received December 28, 2013

The dynamics of solitons in birefringent optical fibers with Kerr law nonlinearity is studied in this paper, in presence of four-wave mixing terms in the governing model. There are three main types of exact one-soliton solutions retrieved for this dynamical system. They are bright, dark, and singular solitons. The ansatz approach is the integration tool for the governing coupled equations. Several constraint conditions naturally fall out during the course of integration of the corresponding coupled nonlinear partial differential equations.

Key words: Kerr law; solitons; integrability; birefringent fibers; four-wave mixing.

PACS: 42.65.-k, 42.65.Ky, 42.65.Tg.

1. INTRODUCTION

Solitons, *i.e.*, self-sustained localized structures, are among the most fascinating nonlinear wave phenomena in nature [1]-[34], and optical solitons are among the most intriguing phenomena in nonlinear optics, due to their emerging applications in optical switching and all-optical processing of information; see a comprehensive book on optical solitons [1] and a few relevant review articles published during the past several years [2–5].

The study of optical solitons in birefringent fibers is an important area of research in nonlinear optics. Solitons propagating through optical fibers get polarized into two pulses due to fiber non-uniformities and other technical factors that arise from fiber technologies. These lead to several issues such as differential group delay, polarization mode dispersion, and many others [6]. In addition to the usual group

Rom. Journ. Phys., Vol. 59, Nos. 5-6, P. 582–589, Bucharest, 2014

velocity dispersion (GVD), an additional dispersion term is taken into consideration in this paper. This is the spatio-temporal dispersion (STD). The inclusion of STD in the governing nonlinear Schrödinger's equation (NLSE) makes it into a well-posed problem [10]. This paper will study optical solitons in birefringent fibers with Kerr law nonlinearity in presence of four-wave mixing phenomenon. Exact soliton solutions will be obtained with several constraint conditions that must remain valid in order for such solitons to exist. Bright, dark, and singular soliton solutions will be reported. The *ansatz method* will be the integration tool of the corresponding coupled nonlinear partial differential equations.

2. MATHEMATICAL MODEL

Optical solitons in birefringent fibers with Kerr law nonlinearity is governed by the following coupled NLSE [6]:

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + \left(\xi_1 |q|^2 + \eta_1 |r|^2 \right) q + i\alpha_1 q_x + \beta_1 q + \sigma_1 q^* r^2 = 0 \quad (1)$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + \left(\xi_2 |q|^2 + \eta_2 |r|^2 \right) r + i\alpha_2 r_x + \beta_2 r + \sigma_2 r^* q^2 = 0 \quad (2)$$

This coupled system of NLSEs, given by Eqs. (1) and (2) governs soliton propagation through nonlinear optical fibers with Kerr law nonlinearity. Here a_l for $l = 1, 2$ are coefficients of GVD while b_l represents coefficients of STD. Then ξ_l and η_l are the coefficients of self-phase modulation (SPM) and cross-phase modulation (XPM), respectively; α_l is proportional to inverse group velocity difference and β_l are proportional to difference between propagation constants. Finally, σ_l gives the four-wave mixing terms.

Equations (1) and (2) are going to be solved for bright, dark, and singular solitons in this paper. The integration tool that will be adopted is the *ansatz* approach. This integration architecture will lead to exact soliton solution and the corresponding relevant constraint conditions will naturally fall out during the integration process. First, a general hypothesis of the solution structure is assumed. Subsequently, the individual types of solitons will be obtained in the next sections.

For integrability aspects of this coupled NLSE by *ansatz method*, an assumption of the following form is considered [6, 9]:

$$q(x, t) = P_1(x, t)e^{i\phi(x, t)} \quad (3)$$

$$r(x, t) = P_2(x, t)e^{i\phi(x, t)} \quad (4)$$

where $P_l(x, t)$ ($l = 1, 2$) are the amplitude components of the soliton solution while the phase component $\phi(x, t)$ is given by

$$\phi(x, t) = -\kappa x + \omega t + \theta. \quad (5)$$

Here, κ is the frequency of the solitons while ω represents the wave number and θ is the phase constant. Substituting (3)-(5) into (1) and (2) and then decomposing into real and imaginary parts give

$$a_l \frac{\partial^2 P_l}{\partial x^2} + b_l \frac{\partial^2 P_l}{\partial x \partial t} + P_l (b_l \omega \kappa - \omega - a_l \kappa^2 + \alpha_l \kappa + \beta_l) + \xi_l P_l^3 + (\eta_l + \sigma_l) P_l P_l^2 = 0 \quad (6)$$

and

$$(1 - b_l \kappa) \frac{\partial P_l}{\partial t} + (b_l \omega - 2a_l \kappa + \alpha_l) \frac{\partial P_l}{\partial x} = 0, \quad (7)$$

respectively. Here, $l = 1, 2$ and $\bar{l} = 3 - l$. From the imaginary part equation it is possible to obtain the speed (v) of the soliton as

$$v = \frac{2a_l \kappa - b_l \omega - \alpha_l}{b_l \kappa - 1} \quad (8)$$

since $P_l(x, t)$ can be represented as $g(x - vt)$, where the function g is the soliton wave profile depending on the type of nonlinearity and v is the speed of the soliton. Now, equating the two values of the soliton speed, from (8), leads to

$$2(a_1 b_2 - a_2 b_1) \kappa^2 - \kappa \{(\alpha_1 b_2 - \alpha_2 b_1) + 2(a_1 - a_2)\} + (b_1 - b_2) \omega + (\alpha_1 - \alpha_2) = 0 \quad (9)$$

From (9), the coefficients of linearly independent functions imply

$$a_1 = a_2, \quad (10)$$

$$b_1 = b_2 \quad (11)$$

and

$$\alpha_1 = \alpha_2. \quad (12)$$

The speed of the soliton therefore is

$$v = \frac{2a\kappa - b\omega - \alpha}{b\kappa - 1} \quad (13)$$

where it is assumed that $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $\alpha_1 = \alpha_2 = \alpha$. The speed of the soliton, given by (13) stays valid as long as

$$b\kappa \neq 1. \quad (14)$$

Therefore, the coupled NLSE in optical fibers simplifies to

$$iq_t + aq_{xx} + bq_{xt} + (\xi_1 |q|^2 + \eta_1 |r|^2) q + i\alpha q_x + \beta_1 q + \sigma_1 q^* r^2 = 0 \quad (15)$$

$$ir_t + ar_{xx} + br_{xt} + (\xi_2 |q|^2 + \eta_2 |r|^2) r + i\alpha r_x + \beta_2 r + \sigma_2 r^* q^2 = 0 \quad (16)$$

It is this coupled NLSE in birefringent fibers that will be analyzed further in the next sections.

3. FAMILIES OF SOLITON SOLUTIONS

3.1. BRIGHT SOLITONS

For bright solitons, one assumes [6]

$$P_l = A_l \operatorname{sech}^{p_l} \tau, \tag{17}$$

where

$$\tau = B(x - vt). \tag{18}$$

Here, A_l represents the soliton amplitude and B is the inverse width of the soliton. Substituting (17) into (6) gives

$$\begin{aligned} & p_l(p_l + 1) A_l B^2 (bv - a) \operatorname{sech}^{p_l+2} \tau \\ & - \{ p_l^2 A_l B^2 (bv - a) + A_l (\omega + a\kappa^2 - b\kappa\omega - \alpha\kappa - \beta_l) \} \operatorname{sech}^{p_l} \tau \\ & + \xi_l A_l^3 \operatorname{sech}^{3p_l} \tau + (\eta_l + \sigma_l) A_l A_l^2 \operatorname{sech}^{p_l+2p_l} \tau = 0 \end{aligned}$$

Balancing principle yields

$$p_l + 2 = 3p_l = p_l + 2p_l \tag{19}$$

so that

$$p_l = 1 \tag{20}$$

for $l = 1, 2$. Next, from (19), setting the coefficients of the linearly independent functions $\operatorname{sech}^{p_l+j} \tau$ to zero, for $j = 0, 2$ leads to the speed and wave number of the soliton as

$$v = \frac{2aB^2 - \xi_1 A_1^2 - (\eta_1 + \sigma_1) A_2^2}{2bB^2} \tag{21}$$

and

$$v = \frac{2aB^2 - \eta_2 A_2^2 - (\xi_2 + \sigma_2) A_1^2}{2bB^2} \tag{22}$$

while the wave numbers are given by

$$\omega = \frac{2a\kappa^2 - 2\alpha\kappa - 2\beta_1 - \xi_1 A_1^2 - (\eta_1 + \sigma_1) A_2^2}{2(b\kappa - 1)}. \tag{23}$$

and

$$\omega = \frac{2a\kappa^2 - 2\alpha\kappa - 2\beta_2 - \eta_2 A_2^2 - (\xi_2 + \sigma_2) A_1^2}{2(b\kappa - 1)}. \tag{24}$$

Equating the expressions for soliton speed v from (21) and (22) implies

$$\frac{A_1}{A_2} = \sqrt{\frac{\eta_2 - \eta_1 - \sigma_1}{\xi_1 - \xi_2 - \sigma_2}} \tag{25}$$

with the condition

$$(\eta_2 - \eta_1 - \sigma_1)(\xi_1 - \xi_2 - \sigma_2) > 0. \tag{26}$$

Finally, equating the two expressions for the soliton wave numbers from (23) and (24) yields

$$2(\beta_2 - \beta_1) + A_1^2(\xi_2 - \xi_1 - \sigma_2) + A_2^2(\eta_2 - \eta_1 - \sigma_1) = 0. \quad (27)$$

Equations (26) and (27) together imply

$$\beta_1 = \beta_2. \quad (28)$$

Therefore, the coupled NLSEs given by (15) and (16) further reduce to

$$iq_t + aq_{xx} + bq_{xt} + \left(\xi_1 |q|^2 + \eta_1 |r|^2\right)q + i\alpha q_x + \beta q + \sigma_1 q^* r^2 = 0 \quad (29)$$

$$ir_t + ar_{xx} + br_{xt} + \left(\xi_2 |q|^2 + \eta_2 |r|^2\right)r + i\alpha r_x + \beta r + \sigma_2 r^* q^2 = 0 \quad (30)$$

where it is again assumed that $\beta_1 = \beta_2 = \beta$. Therefore, the bright one-soliton solution in birefringent fibers is given by

$$q(x, t) = A_1 \operatorname{sech}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)} \quad (31)$$

$$r(x, t) = A_2 \operatorname{sech}[B(x - vt)]e^{i(-\kappa x + \omega t + \theta)}. \quad (32)$$

These bright solitons will exist provided the constraint condition holds:

$$b\kappa \neq 1. \quad (33)$$

3.2. DARK SOLITONS

For dark solitons, the starting hypothesis is given by [6]:

$$P_l = A_l \tanh^{p_l} \tau \quad (34)$$

with the definition of τ being the same as in (18). However for dark solitons the parameters A_l and B are free parameters. Substituting (34) into (6) leads to

$$\begin{aligned} & A_1 B^2 p_l (p_l + 1) (a - bv) \tanh^{p_l + 2} \tau \\ & - \{2A_l B^2 p_l^2 (a - bv) - A_l (\omega + a\kappa^2 - b\kappa\omega - \alpha\kappa - \beta_l)\} \tanh^{p_l} \tau \\ & + A_l B^2 p_l (p_l - 1) (a - bv) \tanh^{p_l - 2} \tau + \xi_l A_l^3 \tanh^{3p_l} \tau \\ & + \xi_l A_l^3 \tanh^{3p_l} \tau + (\eta_l + \sigma_l) \tanh^{p_l + 2p_l} \tau = 0 \end{aligned} \quad (35)$$

By balancing principle, one reaches the same values of the exponents p_l as in (20). Moreover, the standalone linearly independent functions $\tanh^{p_l - 2} \tau$ also yields this same value of p_l . Next, setting the coefficients of other linearly independent functions $\tanh^{p_l + j} \tau$ to zero, for $j = 0, 2$ gives

$$v = \frac{2aB^2 + \xi_1 A_1^2 + (\eta_1 + \sigma_1) A_2^2}{2bB^2} \quad (36)$$

and

$$v = \frac{2aB^2 + \eta_2 A_2^2 + (x i_2 + \sigma_2) A_1^2}{2bB^2}, \tag{37}$$

and the wave numbers are revealed as

$$\omega = \frac{a\kappa^2 - \alpha\kappa - \beta_1 - \xi_1 A_1^2 - (\eta_1 + \sigma_1) A_2^2}{b\kappa - 1}. \tag{38}$$

and

$$\omega = \frac{a\kappa^2 - \alpha\kappa - \beta_2 - \eta_2 A_2^2 - (\xi_2 + \sigma_2) A_1^2}{b\kappa - 1}. \tag{39}$$

From (38) and (39), once again recovers (28) as in the case of bright solitons. Also from (36) and (37) one recovers (25) and (26). This leads to the dark one-soliton solution as

$$q(x, t) = A_1 \tanh[B(x - vt)] e^{i(-\kappa x + \omega t + \theta)} \tag{40}$$

$$r(x, t) = A_2 \tanh[B(x - vt)] e^{i(-\kappa x + \omega t + \theta)}. \tag{41}$$

which remain valid for the domain restrictions as indicated above.

3.3. SINGULAR SOLITONS

For singular solitons, the starting hypothesis is given by [18]:

$$P_l = A_l \operatorname{csch}^{p_l} \tau \tag{42}$$

where parameters A_l and B are again free parameters. Upon substituting (42) into (6) gives

$$\begin{aligned} & A_l B^2 p_l (p_l + 1) (a - bv) \operatorname{csch}^{p_l + 2} \tau \\ & + \{ A_l B^2 p_l^2 (a - bv) - A_l (\omega + a\kappa^2 - b\kappa\omega - \alpha\kappa - \beta_l) \} \operatorname{csch}^{p_l} \tau \\ & + \xi_l A_l^3 \operatorname{csch}^{3p_l} \tau + (\eta_l + \sigma_l) A_l A_l^2 \operatorname{csch}^{p_l + 2p_l} \tau = 0 \end{aligned} \tag{43}$$

Balancing principle gives the same value of p_l as in (20). Next, from (43), setting the coefficients of the linearly independent functions $\operatorname{csch}^{p_l + j} \tau$ to zero, for $j = 0, 2$ leads to the speed and wave number of the soliton as

$$v = \frac{2aB^2 + \xi_1 A_1^2 + (\eta_1 + \sigma_1) A_2^2}{2bB^2} \tag{44}$$

and

$$v = \frac{2aB^2 + \eta_2 A_2^2 + (\xi_2 + \sigma_2) A_1^2}{2bB^2} \tag{45}$$

while the wave numbers are

$$\omega = \frac{2a\kappa^2 - 2\alpha\kappa - 2\beta_1 + \xi_1 A_1^2 + (\eta_1 + \sigma_1) A_2^2}{2(b\kappa - 1)}. \tag{46}$$

and

$$\omega = \frac{2a\kappa^2 - 2\alpha\kappa - 2\beta_2 + \eta_2 A_2^2 + (\xi_2 + \sigma_2) A_1^2}{2(b\kappa - 1)}. \quad (47)$$

Equating the values of the speed and wave numbers from their respective components again yields (25), (26), and (28). The singular one-soliton solution for birefringent fibers is

$$q(x, t) = A_1 \operatorname{csch}[B(x - vt)] e^{i(-\kappa x + \omega t + \theta)} \quad (48)$$

$$r(x, t) = A_2 \operatorname{csch}[B(x - vt)] e^{i(-\kappa x + \omega t + \theta)}. \quad (49)$$

with their respective constraint conditions as indicated above.

4. CONCLUSIONS

This paper studies the dynamics of optical solitons in birefringent fibers with Kerr law nonlinearity in presence of four-wave mixing. Exact bright, dark, and singular one-soliton solutions are obtained for birefringent fibers, in presence of four-wave mixing terms in the governing coupled nonlinear evolution equations. There are constraint conditions that must hold in order for the solitons to exist. These results will lead to several additional integrability issues when Hamiltonian perturbation terms are taken into consideration. Those results will be reported elsewhere. Additionally, in future, these results could be extended to dense wavelength division multiplexing physical settings.

Acknowledgements. This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University under grant number (14-130/1433 HiCi). The authors therefore acknowledge technical and financial support of KAU.

REFERENCES

1. Y.S. Kivshar and G.P. Agrawal, *Optical Solitons: From Fibers to Photonics Crystals* (Academic Press, San Diego, 2003).
2. Z. Chen, M. Segev, and D.N. Christodoulides, *Rep. Prog. Phys.* **75**, 086401 (2012).
3. P. Grelu and N. Akhmediev, *Nature Photon.* **6**, 84 (2012).
4. Y.V. Kartashov, B.A. Malomed, and L. Torner, *Rev. Mod. Phys.* **83**, 247 (2011).
5. B.A. Malomed *et al.*, *J. Opt. B: Quantum Semiclassical Opt.* **7**, R53 (2005).
6. A. Biswas, K.R. Khan, A. Rahman, A. Yildirim, T. Hayat, and O.M. Aldossary, *J. Optoelectron. Adv. Mat.* **14**, 571 (2012).
7. S. Coen, A.H. L. Chau, R. Leohardt, J.D. Harvey, J.C. Knight, W.J. Wadsworth, and P.St.J. Russell, *J. Opt. Soc. Am. B* **19**, 753 (2002).
8. D. Grecu, A. Visinescu, R. Fedele, and S. De Nicola, *Rom. J. Phys.* **55**, 585 (2010).
9. L. Girgis, D. Milovic, S. Konar, A. Yildirim, H. Jafari, and A. Biswas, *Rom. Rep. Phys.* **64**, 663 (2012).

10. S. Kumar, K. Singh, and R.K. Gupta, *Pramana* **79**, 41 (2012).
11. D. Mihalache *et al.*, *Phys. Rev. E* **62**, 7340 (2000).
12. D. Mihalache, D. Mazilu, L.-C. Crasovan, and L. Torner, *Opt. Commun.* **137**, 113 (1997).
13. D. Mihalache, F. Lederer, D. Mazilu, and L.-C. Crasovan, *Opt. Eng.* **35**, 1616 (1996).
14. D. Mihalache *et al.*, *Phys. Rev. A* **77**, 033817 (2008).
15. I.V. Mel'nikov, D. Mihalache, and N.-C Panoiu, *Opt. Commun.* **181**, 345 (2000).
16. D. Mihalache, *Proc. Romanian Acad. A* **11**, 142 (2010).
17. D. Milovic and A. Biswas, *Serbian Journal of Electrical Engineering* **10**, 365 (2013).
18. M. Savescu, K.R. Khan, P. Naruka, H. Jafari, L. Moraru, and A. Biswas, *Journal of Computational and Theoretical Nanoscience* **10**, 1182 (2013).
19. Q.-B. Xu and Q.-S. Chang, *Acta Mathematicae Applicatae Sinica, English Series* **26**, 205 (2010).
20. A.H. Bhrawy, M.A. Abdelkawy, S. Kumar, and A. Biswas, *Rom. J. Phys.* **58**, 729 (2013).
21. W. Wang, Y. Sheng, X. Niu, M. Huang, S. Zheng, and Y. Kong, *Optics & Laser Technology*, **58**, 16 (2014).
22. W. Qian, C.L. Zhao, X. Dong, and W. Jin, *Opt. Commun.* **283**, 5250 (2010).
23. B.S. Ahmed and A. Biswas, *Proc. Romanian Acad. A* **14**, 111 (2013).
24. Guangye Yang *et al.*, *Rom. Rep. Phys.* **65**, 902 (2013).
25. Guangye Yang *et al.*, *Rom. Rep. Phys.* **65**, 391 (2013).
26. A. Salas, S. Kumar, A. Yildirim, and A. Biswas, *Proc. Romanian Acad. A* **14**, 28 (2013).
27. H. Triki *et al.*, *Rom. J. Phys.* **57**, 1029 (2012).
28. Y.J. He and D. Mihalache, *Rom. Rep. Phys.* **64**, 1243 (2012).
29. D. Mihalache, *Rom. J. Phys.* **57**, 352 (2012).
30. H. Leblond, H. Triki, and D. Mihalache, *Rom. Rep. Phys.* **65**, 925 (2013).
31. H. Leblond and D. Mihalache, *Phys. Reports* **523**, 61 (2013).
32. Zai-Yun Zhang *et al.*, *Rom. J. Phys.* **58**, 766 (2013).
33. G. Ebadi *et al.*, *Rom. J. Phys.* **58**, 3 (2013).
34. A.M. Wazwaz, *Rom. J. Phys.* **58**, 685 (2013).

Nonlinear optics facilitates interaction of photons with each other and with other physical systems, and is of prime importance in quantum optics. Remarkably, the ordinary and extraordinary refractive coefficients in birefringent materials can significantly differ. This allows for the differential tuning of the TE and TM WGM spectra, a capability very important for some sensor applications [211, 212] and particularly for achieving the phase matching in various non-linear optics applications [213]. This lesson demonstrates that the Kerr nonlinearity, which stabilizes solitons against spreading due to GVD, also stabilizes them against splitting due to birefringence [1] [3]. The "single-mode" fibers are actually bimodal because of the presence of birefringence. This means that the group velocity is different for pulses polarized along the two principal axes. If the pulse contains both polarization components, in addition to spreading due to GVD, the partial pulses will tend to split apart because of birefringence [1], [2]. In the anomalous GVD regime, the Kerr nonlinearity can compensate. Optical solitons in birefringent fibers with parabolic law nonlinearity is governed by the following coupled NLSE [25]. This paper analyzed optical solitons in birefringent fibers with 4WM for Kerr and parabolic laws of nonlinearity. The extended trial function approach retrieved bright and singular soliton solutions along with several other forms of waves that includes periodic singular waves and other solutions. These solutions appeared with different constraint conditions that guarantees the existence of the variety of waves. The phase-matching condition enables the extraction of these waves for birefringent fibers with 4WM.