Book Review

C. J. Ash and J. Knight. ***Computable Structures and the Hyperarithmetical Hierarchy.*** Studies in Logic and the Foundations of Mathematics, vol. 144. Elsevier, Amsterdam etc. 2000, xv + 346 pp.

Valentina Harizanov
Department of Mathematics
The George Washington University
Washington, D.C., 20052, USA
E-mail: harizanv@gwu.edu

Chris Ash started *Computable Structures and the Hyperarithmetical Hierarchy* to bring together, in a consistent and compact form, results from computable (recursive, effective) model theory. Much of this work grew out of his collaboration with Julia Knight over the decade of 1985-95. Tragically, Chris Ash's life came to a sudden end in 1995. Julia Knight, working from Ash's outline and pieces of the first five chapters, succeeded in completing this beautiful book.

The book fills an important gap. Although there have been a few Russian monographs, this is the first book from the English-speaking world to cover computable model theory. It generally follows the Western framework of computable mathematics, but includes many Russian results. It focuses on the correspondence between *definability* (syntactic properties) and *computability* (semantic properties) of phenomena in algebraic structures. Naturally, syntactic conditions themselves often involve some form of algorithmic definability which accounts for limitations on complexity on computable isomorphic structures. Ash-Knight program can be traced back to Ash-Nerode work in 1981.

This is a book about computable and non-computable properties of structures. The structures studied are countable and their languages are computable. Such a structure with a computable domain is *computable* if its atomic diagram is computable. If, moreover, the complete diagram of the structure is computable, then the structure is called *decidable.* While the standard model of arithmetic is computable, its theory, called *true arithmetic*, is not even arithmetical. In addition to complexity of structures and their theories, algorithmic properties studied include complexity of additional relations, and isomorphisms of structures.

The book is self-contained. It has nineteen chapters and an extensive bibliography. The book begins with a survey of fundamental results from computability theory on natural numbers, and from classical model theory. The computational difficulty of structures, their relations and isomorphisms is measured within the *hyperarithmetical hierarchy*, introduced by Kleene in the 1950's. In this hierarchy, computational steps are iterated beyond natural numbers by using constructive ordinals, whose notations are recursively defined. *Constructive ordinals* coincide with *computable ordinals*, that is, with the order types of computable well orderings. Constructive ordinals can also be used to assign computability-syntactic complexity to infinitary first-order formulae. The material on constructive ordinals and computable formulae is presented in detail, in an elegant way, suitable for applications to effective model theory.

In the first part of the book, which includes the first eight chapters, the authors develop tools necessary for the study of computational difficulty of problems arising on algebraic structures. The book starts with a survey of basic computability theory, including the *s-m-n* theorem, the recursion theorem, computably enumerable sets, Turing reducibility, and arithmetical hierarchy. Then the basic model theory of first-order languages follows, including elementary substructures, consistency criteria, compactness, and prime, saturated and homogeneous models of theories. The authors further discuss computability-theoretic complexity of theories and structures, and prove Tennenbaum's theorem that there is no computable non-standard model of Peano arithmetic. They then introduce Kleene's ordinal notation and develop the theory of constructive ordinals. They also describe the analytical hierarchy and show that hyperarithmetical sets coincide with Δ^1_1 sets. For a countable language L, the authors define (infinitary) formulae of $L_{\omega_1\omega}$ and their computable analogs. *Computable infinitary formulae* were introduced by Ash in 1986. The authors further present the Makkai consistency criterion, and Scott's isomorphism theorem. They prove a variant of Barwise compactness theorem and give several of its applications. This part of the book can be used in a course on hyperarithmetical hierarchy.

The middle part of the book, which consists of Chapters 9-12, begins with a series of results on the existence of computable structures of well-known theories. For example, Khisamiev gave a partial characterization of the reduced Abelian p-groups with computable isomorphic copies, using Ulm invariants. Goncharov and Peretyat'kin independently characterized when a countable homogeneous structure has a decidable isomorphic copy. The book further investigates computability-theoretic complexity of images of a new relation R on the domain of computable structure $\mathcal{A}$ under isomorphisms from $\mathcal{A}$ to other computable structures. If all such images of R belong to a complexity class $\mathcal{P}$ then R is called *intrinsically* $\mathcal{P}$ *on* $\mathcal{A}$. In the case of non-computable countable structures, we can investigate the corresponding relative versions of algorithmic properties. These relative results are usually simpler and involve the forcing technique, while the results on computable copies often use the priority method. The final chapter in this part (Chapter 12) studies the important notions of computable stability and computable categoricity. A computable structure is *computably stable* if every isomorphism onto another computable structure is computable. Ash and Nerode and independently Goncharov syntactically characterized computable stability. Goncharov also gave a syntactic characterization of computable categoricity. A computable structure $\mathcal{A}$ is *computably categorical* if for every isomorphic computable structure $\mathcal{B}$, there is at least one computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. In this case, we also say that $\mathcal{A}$ and $\mathcal{B}$ have the same computable isomorphism type, or that $\mathcal{A}$ has computable dimension 1. *Computable dimension* of a countable structure is the number of its computable isomorphism types. Computable dimension of familiar mathematical structures, such as linear orderings, Boolean algebras and Abelian groups, is often 1 or ω. Goncharov constructed, however, a structure of any finite computable dimension.

The last part of the book is the most advanced and complex. It consists of Chapters 13-19. It starts with a gradual exposition of a general, powerful, and intricate machinery for nested priority constructions, invented by Ash in the 1980's and further developed by Ash and Knight and their students. This method is suitable for many computable model-theoretic priority constructions at various levels of the hyperarithmetical hierarchy. The method is formulated in terms of Ash's so-called α*-systems*, where α is a computable ordinal. The authors prove the corresponding abstract metatheorems whose conditions guarantee the success of desired constructions. They further present a number of applications of metatheorems. These include Barker's result for intrinsically Σ^0_α relations, and Davey's result on inseparability of relations on computable structures. They include Ash's characterizations of stability and categoricity of computable structures in hyperarithmetical degrees, and Ash and Knight's result on pairs of computable structures. These results often require constructions of isomorphisms which exploit the *back-and-forth relations* between finite sequences of elements of structures. The last chapter of the book describes results on complexity of *models of arithmetic.* It presents Harrington's theorem on the existence of a non-standard arithmetical model of Peano arithmetic, whose theory is not arithmetical.

Harrington's theorem, for which he developed his workers method, is proved using an α-system. This chapter also presents Solovay and Marker's characterization of the degrees of non-standard models of true arithmetic as the degrees of enumerations of Scott sets containing the arithmetical sets.

This book is a valuable reference for every logician. The first two parts can also be used in a computable model theory course. In addition to general structures, particular structures whose various algorithmic properties are investigated include equivalence structures, trees, linear and well-orderings, Boolean algebras, models of arithmetic, Abelian groups, fields, and vector spaces. The last part of the book is more advanced and involves general techniques used in current research. The book is well written, with proofs that are intuitively motivated, elegant, and to the point. You will truly enjoy reading it.