A Note on Modified Gabor Frames

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Abstract

In this paper we generalize a procedure, originally proposed by Kaiser, which produces a family of (A, B)-frames in $\mathcal{L}^2(\boldsymbol{R})$, starting from a given Gabor (A, B)-frame. The procedure is applied to several examples.

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1 Introduction

In 1994 Kaiser has shown how, given an $(A,B)-$frame $\mathcal{F}$, it is possible to construct a different frame, with the same bounds A and B, which is related to $\mathcal{F}$ by means of an unitary operator $W(\theta)=\exp\{i\theta H\}$, $\theta\in\boldsymbol{R}$. In Kaiser's work H was nothing but the (self-adjoint) hamiltonian of an harmonic oscillator, [1] and the procedure was constructed to work for Gabor frames. So, without major changes, it is obvious that Kaiser's idea only makes sense in the space of the square integrable functions, $\mathcal{L}^2(\boldsymbol{R})$.

Something along the same lines has been proposed, still in $\mathcal{L}^2(\boldsymbol{R})$, by Baraniuk and Jones in reference [2], where it is introduced a different unitary operator which maps orthonormal bases into orthonormal bases (and frames into frames). Some comments on the possibility of modifying frames in arbitrary Hilbert spaces are contained in Kaiser's book [3], while a detailed approach to this same subject from a different point of view is contained in reference [4].

In this note we want to show how Kaiser's original approach can be generalized to several operators H and what need to be changed, working only in $\mathcal{H}=\mathcal{L}^2(\boldsymbol{R})$.

This work is organized as follows:

in the next short Section we remind some standard definitions and results about frames and we give the main outcome contained in reference [1];

in Section 3 we propose our generalized version of Kaiser's approach, discussing in details many examples. In particular, we show what happens if we use as a starting frame a family of overcomplete coherent states. We conclude the Section, and the paper, with our plans for the future.

2 Useful Results about Frames

Given a set of indexes $\mathcal{I}$ and two constants $0<A\leq B<\infty$, we say that $\{\varphi_i\in\mathcal{H}:\, i\in\mathcal{I}\}$ is an $(A,B)-$frame in the Hilbert space $\mathcal{H}$ if the following inequalities

$$A\|f\|^2\leq\sum_{i\in\mathcal{I}}|<\varphi_i,f>|^2\leq B\|f\|^2 \tag{2.1}$$

hold for all $f\in\mathcal{H}$. The general theory of frames is rather well-developed, see [5] for a review, and the utility of this set of vectors is very clear: the φ_i's span the whole Hilbert space but form, in general, an overcomplete set. This implies that any vector of $\mathcal{H}$ can be expanded in terms of the vectors of the frame, but also that this expansion is not unique. In reference [5] we see how it is possible to associate to a given frame a bounded operator F from $\mathcal{H}$ into the space of the square-summable sequences, $l^2(\mathcal{I})$, and how to use this operator, together with its adjoint F^*, to construct the so-called dual frame $\{\tilde{\varphi}_i \in \mathcal{H} : \; i \in \mathcal{I}\}$. Together, the frame and its dual give the following expansions:

$$f = \sum_{i\in\mathcal{I}} < \varphi_i, f > \tilde{\varphi}_i = \sum_{i\in\mathcal{I}} < \tilde{\varphi}_i, f > \varphi_i,$$

which hold for any f in the Hilbert space $\mathcal{H}$, [5].

In this paper we will consider only a very particular situation, which has a particular interest since it is related to signal analysis, wavelets, and all this staff: first of all we take $\mathcal{L}^2(\boldsymbol{R})$ to be our Hilbert space. Moreover, the frames we consider are only Gabor frames. This means the following: let q and p be the usual position and momentum operators in $\mathcal{L}^2(\boldsymbol{R})$, satisfying $qp - pq = i1\!\!1$. They are essentially self-adjoint, so that the operators

$$U(\omega) \equiv \exp\{i\omega q\}, \qquad\qquad V(s) \equiv \exp\{-isp\} \tag{2.2}$$

are unitary. Here we have used the same notation to indicate both the operators q and p and their extensions. Without considering the details of the Weyl-Heisenberg group generated by these operators, we know that, given a window function $g(t) \in \mathcal{L}^2(\boldsymbol{R})$, $\|g\| = 1$, it is possible to *translate* and *modulate* $g(t)$ by means of $U(\omega)$ and $V(s)$ in the following way

$$g_{\omega,s} = U(\omega)V(s)g.$$

This can also be written as $g_{\omega,s}(t) = \exp\{i\omega t\}g(t-s)$. How it is well known, under given assumptions on the function $g(t)$, it is possible to extract a discrete subset of $\boldsymbol{R}^2$, Γ, such that the set $\mathcal{F} \equiv \{g_{\omega,s}, (\omega, s) \in \Gamma\}$ is an $(A, B)-$frame. It is evident that the existence of this discrete set Γ reminds very closely the analogous existence of a discrete lattice in the phase space which makes of a (generally) overcomplete set of coherent states, [6], a complete set, [7, 8].

In a series of papers, [9, 10], Ali and others proposed a generalized version of frames, the so-called continuous frames (CF), in which Γ is not necessarily a discrete set of indexes. For these CF the sum in (2.1) should be replaced by an integral (with a certain measure). A typical example of CF, which will be used in the next Section, is the family of the following coherent states $g_{\omega,s}(t) = \frac{1}{\pi^{1/4}} e^{i\omega t} e^{-(t-s)^2/2}$, with $(\omega, s) \in \boldsymbol{R}^2$.

In reference [1] it has been discussed the possibility of obtaining a different $(A, B)-$frame starting from a given one, $\mathcal{F}$, generated by a given square integrable function g. The procedure is summarized by the following steps:

- we introduce the self-adjoint operator on $\mathcal{L}^2(\boldsymbol{R})$, $H = \frac{1}{2}(q^2 + p^2)$, and the corresponding unitary operator $W(\theta) = e^{i\theta H}$;

- we use $W(\theta)$ to rotate q and p: $q(\theta) \equiv W(\theta)qW(-\theta) = q\cos(\theta) + p\sin(\theta)$ and $p(\theta) \equiv W(\theta)pW(-\theta) = p\cos(\theta) - q\sin(\theta)$;

- we observe that $W(\theta)g_{\omega,s} = e^{i\omega q(\theta)}e^{-isp(\theta)}W(\theta)g$;

- calling now $g^\theta \equiv W(\theta)g$, which is still in $\mathcal{L}^2(\boldsymbol{R})$ due to the unitarity of W, and defining the real functions $s(\theta) = s\cos(\theta) - \omega\sin(\theta)$ and $\omega(\theta) = \omega\cos(\theta) + s\sin(\theta)$, we find that

$$W(\theta)g_{\omega,s} = \gamma(\omega, s, \theta)g^\theta_{\omega(\theta),s(\theta)}, \tag{2.3}$$

  where γ is a phase.

- Finally, calling $\Gamma_\theta = \{(\omega(\theta), s(\theta)) \in \boldsymbol{R}^2 : (\omega, s) \in \Gamma\}$ and $\mathcal{F}_\theta \equiv \{g^\theta_{\omega,s} : (\omega, s) \in \Gamma_\theta\}$, we deduce that $\mathcal{F}_\theta$ is an $(A, B)-$frame. In this deduction it is essential that the set $\mathcal{F}$ is an $(A, B)-$frame by itself.

This is the main contain of reference [1]: a Gabor frame generates, by means of the operator H above, a family of Gabor frames, all with the same frame bounds. This is clearly quite different from what the author made in a previous work, [4]. In reference [4], in fact, we have shown how it is possible to obtain frames starting from a given frame not necessarily of the Gabor type. Moreover, we had no need to restrict our procedure to $\mathcal{L}^2(\boldsymbol{R})$, since it can be applied in any Hilbert space. Another major difference between the approaches in refs. [1] and [4] is that the frame bounds are, in general, modified by the procedure proposed in [4] so that, for instance, a tight frame can be obtained even if the original frame bounds are different.

3 Generalization of the Kaiser Approach

In this Section we will generalize the schedule given in the previous Section. In particular, we will show that it is possible to use self-adjoint operators different from $\frac{1}{2}(p^2 + q^2)$ to modify a given Gabor frame. Therefore, after a theoretical preamble on the generalized procedure, we will discuss in some detail many examples and applications to coherent states.

If we consider carefully Kaiser's approach, we see that the main ingredient required to construct what we will call *modified Gabor frames*, MGF, is the hamiltonian $H = \frac{1}{2}(p^2 + q^2)$. With this in mind, we give now the following

Criterion:– let us consider a self-adjoint operator h, and its related unitary exponential $W(\theta) \equiv e^{ih\theta}$, $\theta \in \boldsymbol{R}$ for which there exist:

1) two real functions $\omega(\theta)$ and $s(\theta)$ satisfying $\omega(0) = \omega$ and $s(0) = s$;

2) a phase $\Phi(\omega, s, \theta)$ and

3) a unitary operator $T(\omega, s, \theta)$,

such that, calling, as in reference [1], $q(\theta) = W(\theta)qW(-\theta)$ and $p(\theta) = W(\theta)pW(-\theta)$, then

$$e^{i\omega q(\theta)}e^{-isp(\theta)} = \Phi(\omega, s, \theta)U(\omega(\theta))V(s(\theta))T(\omega, s, \theta). \tag{3.1}$$

If these conditions hold, defining

$$g^{T,\theta} \equiv T(\omega, s, \theta)W(\theta)g, \tag{3.2}$$

and

$$g_{\omega,s}^{T,\theta} \equiv U(\omega)V(s)g^{T,\theta}, \tag{3.3}$$

we can prove that the set

$$\mathcal{F}_{T,\theta} \equiv \{g_{\omega,s}^{T,\theta} : (\omega, s) \in \Gamma_\theta\} \tag{3.4}$$

is an $(A, B)-$frame if $\mathcal{F}$, by itself, is an $(A, B)-$frame. Here $\mathcal{F}$ and Γ_θ are the same sets already introduced in the previous Section.

Remarks.– (1) First of all, we see that all these requirements are satisfied in Kaiser's approach: $h = H$, $\Phi = \gamma$, $T = 1\!\!1$, $s(\theta) = s\cos(\theta) - \omega\sin(\theta)$ and $\omega(\theta) = \omega\cos(\theta) + s\sin(\theta)$. Of course, this is not the only example for which the Criterion is satisfied. It is possible to find many other examples, and we will discuss several of them in the second part of this Section.

(2) Secondly, we want to stress that h does not need to be the hamiltonian of any real, or fictitious, physical problem. We only need h be (essentially) self-adjoint. In particular, we will consider in the following examples of operators which are not positive defined, so that they cannot be seen as physical hamiltonians. We will also consider operators with a non-quadratic dependence on p, so that they show no evident 'canonical' kinetic part, p^2.

The proof of our claim is rather simple and follows the same lines of the one given in reference [1]. Let $f \in \mathcal{L}^2(\boldsymbol{R})$ and $(\omega(\theta), s(\theta)) \in \Gamma_\theta$. If the above assumptions are satisfied, we have:

$$\begin{aligned} |< g_{\omega(\theta),s(\theta)}^{T,\theta}, f >| &= |< U(\omega(\theta))V(s(\theta))g^{T,\theta}, f >| = |< e^{i\omega q(\theta)}e^{-isp(\theta)}W(\theta)g, f >| = \\ &= |< W(\theta)U(\omega)V(s)g, f >| = |< g_{\omega,s}, f^{(-\theta)} >|. \end{aligned}$$

Let us now notice that $\|f^{(-\theta)}\| = \|W(-\theta)f\| = \|f\|$, which implies, since the set $\{g_{\omega,s} : (\omega, s) \in \Gamma\}$ is an (A, B)-frame, that $A\|f\|^2 \leq \sum_{(\omega,s)\in\Gamma} |< g_{\omega,s}, f^{(-\theta)} >|^2 \leq B\|f\|^2$.

Therefore, we conclude that

$$A\|f\|^2 \leq \sum_{(\omega,s)\in\Gamma_\theta} |< g_{\omega,s}^{T,\theta}, f >|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{L}^2(\boldsymbol{R}), \tag{3.5}$$

which proves that also the set $\mathcal{F}_{T,\theta}$ is an $(A,B)-$frame.

As a matter of fact, the Criterion does not appear easy to be checked. In fact, given a self-adjoint operator h, it is not very clear how to get all the ingredients introduced above. However, it is possible to give at least a class of operators h for which all the functions and operators of the Criterion can be found: any self-adjoint polynomial in q and p of the form $h = \sum_{k=1}^n (c_k q^k + d_k p^k)$ is an operator satisfying all the requirements above. It is possible to show that, for all such h's, we can find $\omega(\theta)$, $s(\theta)$, $\Phi(\omega, s, \theta)$ and $T(\omega, s, \theta)$ satisfying relation (3.1), and prove, as a consequence of this fact, that $\mathcal{F}_{T,\theta}$ is an $(A,B)-$frame. Here c_k and d_k must be chosen in such a way to make h self-adjoint in $\mathcal{L}^2(\boldsymbol{R})$ ($c_k = 0$ for all k, or $d_k = 0$ for all k, or, yet, $c_k = d_k = 1/2\delta_{k,2}$,...are examples of possible choices for the coefficients). This claim can be deduced also as a 'corollary' of the following more general result:

let h be such that $q(\theta) = e^{ih\theta} q e^{-ih\theta} = q\alpha(\theta) + \pi(\theta, p)$ and $p(\theta) = e^{ih\theta} p e^{-ih\theta} = p\beta(\theta) + \chi(\theta, q)$. Here $\alpha(\theta)$ and $\beta(\theta)$ are real functions satisfying $\alpha(0) = \beta(0) = 1$, $\pi(\theta, p)$ is a real polynomial in p and depends on θ in such a way that $\pi(0, p) = 0$ while $\chi(\theta, q)$ is a real polynomial in q and, again, $\chi(0, p) = 0$. π, χ, α and β must all be regular in θ. If $q(\theta)$ and $p(\theta)$ have these expressions, which is surely the case if $h = \sum_{k=1}^n (c_k q^k + d_k p^k)$, then it is possible to show that our Criterion is satisfied. The proof is based on the Baker-Campbell-Hausdorff (BCH) formula in its complete form, [11, 12], which we give here up the order we will use in this paper:

$$e^x e^y = \exp\{x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) - \frac{1}{24}[x,[y,[x,y]]] + ...\}. \tag{3.6}$$

All the contributions not explicitly written contains higher order commutators. A recurrence formula for the next terms can be found in the papers cited above. We don't want to discuss in detail the reason why for $q(\theta)$ and $p(\theta)$ as above the Criterion is satisfied, since these details will be analyzed in the examples. Here we only want to make our claim reasonable. Using the BCH formula (more than once, depending on the degree of π), we can write

$$e^{i\omega q(\theta)} = e^{i\omega\alpha(\theta)q} e^{i\omega\pi(\theta,p)} e^{i\omega\pi_{-1}(\theta,p)},$$

where $\pi_{-1}(\theta, p)$ is a (real) function of θ and is a (real) polynomial in p of degree $n-1$, n being the degree of $\pi(\theta, p)$. Of course, in general, $\pi_{-1}(\theta, p)$ contains a linear contribution in p, which can be extracted from π_{-1} in the following way: $\pi_{-1}(\theta,p)=\tilde{\pi}_{-1}(\theta,p)+\gamma_p(\theta)p$. Both $\tilde{\pi}_{-1}$ and γ_p are real functions.

Analogous considerations can also be repeated for $e^{-isp(\theta)}$. Therefore we get

$$\begin{aligned}e^{i\omega q(\theta)}e^{-isp(\theta)} &= e^{i\omega\alpha(\theta)q}e^{i\omega\pi(\theta,p)}e^{i\omega\tilde{\pi}_{-1}(\theta,p)}e^{ip(\omega\gamma_p(\theta)-s\beta(\theta))}e^{-is\gamma_q(\theta)q}e^{-is\chi(\theta,q)}e^{-is\tilde{\chi}_{-1}(\theta,q)} = \\ &= e^{i\omega\alpha(\theta)q}e^{ip(\omega\gamma_p(\theta)-s\beta(\theta))}e^{i\omega(\pi(\theta,p)+\tilde{\pi}_{-1}(\theta,p))}e^{-is\gamma_q(\theta)q}e^{-is(\chi(\theta,q)+\tilde{\chi}_{-1}(\theta,q))}.\end{aligned}$$

Comparing this formula with equation (3.1), we see that the explicit forms for Φ, $\omega(\theta)$, $s(\theta)$ and T can be obtained by commuting $e^{-is\gamma_q(\theta)q}$ with the two, p-depending, exponentials to its left. The results of these commutators are, again, obtained by means of the BCH formula with a difficulty which increases more and more depending on the degree n of the polynomial $\pi(\theta,p)$.

The second part of this Section is devoted to the analysis of some examples. These are intended to clarify the above general procedure and to show that the Kaiser procedure can be really extended to many other different and inequivalent situations.

Example 1: $h=p^3$

We treat this example in all the detail to clarify the general strategy. The next examples will be discussed only in their main lines.

The differential equations for $q(\theta)=W(\theta)qW(-\theta)$ and $p(\theta)=W(\theta)pW(-\theta)$ are, of course, $q'(\theta)=iW(\theta)[p^3,q]W(-\theta)=3p(\theta)^2$ and $p'(\theta)=iW(\theta)[p^3,p]W(-\theta)=0$, which are solved by $q(\theta)=q+3\theta p^2$ and $p(\theta)=p$. Of course, here we have considered also the obvious boundary conditions $p(0)=p$ and $q(0)=q$. Using the BCH formula we get

$$e^{i\omega q(\theta)}=e^{i\theta\omega^3}e^{i\omega q}e^{3i\theta\omega^2 p}e^{3i\theta\omega p^2}. \tag{3.7}$$

This result easily follows from the following commutation relations: $[x,y]=-6i\theta\omega^2 p$ and $[x,[x,y]]=6i\theta\omega^3 1\!\!1$, while all the other commutators entering in the BCH formula are identically zero. Here we have defined $x=i\omega q$ and $y=3i\omega\theta p^2$. An immediate consequence of equation (3.7) is that

$$e^{i\omega q(\theta)}e^{-isp(\theta)}=e^{i\theta\omega^3}e^{i\omega q}e^{-ip(s-3\theta\omega^2)}e^{3i\theta\omega p^2}.$$

Therefore, comparing this result with formula (3.1), we deduce that the assumptions of the Criterion are satisfied taking $\omega(\theta)=\omega$, $s(\theta)=s-3\theta\omega^2$, $T=e^{3i\omega\theta p^2}$ and $\Phi=e^{i\theta\omega^3}$. At this point we can safely conclude that, if $\mathcal{F}$ is an $(A,B)-$frame, then also $\mathcal{F}_{T,\theta}$ is an $(A,B)-$frame.

Example 2: $h = q^3$

The differential equations for $q(\theta)$ and $p(\theta)$ are, essentially, specular with respect to the ones of the previous example, and their solutions are $q(\theta) = q$ and $p(\theta) = p - 3\theta q^2$. Using the BCH formula we get $e^{-isp(\theta)} = e^{i\theta s^3} e^{-isp} e^{3is^2\theta q} e^{3i\theta s q^2}$, so that

$$e^{i\omega q(\theta)} e^{-isp(\theta)} = e^{-2i\theta s^3} e^{i(\omega+3\theta s^2)q} e^{-ips} e^{3i\theta s q^2},$$

which is obtained using also the familiar consequence of the BCH formula, $e^x e^y = e^y e^x e^{[x,y]}$. This commutation rule holds since $[x, y]$ here is simply a c-number ($x = -isp, y = 3i\theta s^2 q$). Therefore, the assumptions of the Criterion are satisfied once again with $\omega(\theta) = \omega + 3\theta s^2$, $s(\theta) = s$, $T = e^{3is\theta q^2}$ and $\Phi = e^{-2i\theta s^3}$. Again, if the set $\mathcal{F}$ is an $(A, B)-$frame, also $\mathcal{F}_{T,\theta}$ turns out to be an $(A, B)-$frame.

Example 3: $h = q^4$

It is easy to see that $q(\theta) = q$ and $p(\theta) = p - 4\theta q^3$. It is obvious that the expansion of $e^{-is(p-4\theta q^3)}$ presents more difficulties with respect to those found in Examples 1 and 2, the reason being that in the BCH formula we need to consider another term in the expansion. Applying twice the BCH formula we obtain

$$e^{-isp(\theta)} = e^{i\theta s^4} e^{-isp} e^{4i\theta s^3 q} e^{4i\theta s q^3} e^{6i\theta s^2 q^2},$$

which, after minor computations, suggests the following identifications: $\omega(\theta) = \omega + 4\theta s^3$, $s(\theta) = s$, $T = e^{i\theta(4sq^3+6s^2q^2)}$ and $\Phi = e^{-3i\theta s^4}$. This shows that also this example satisfies all the requirements of the Criterion, so that also the operator $h = q^4$ returns a MGF, following the procedure described here.

Remark:– Analogous computations can be repeated for $h = p^4$, obtaining, again, similar conclusions. It appears evident that, paying the price of increasing the difficulty of the computations, it is possible to consider self-adjoint operators of the form $h = q^n$ or $h = p^n$, for some integer n. Of course the choice $n = 1$ is not very interesting, while $n = 2$, which we have not considered in detail here, is already non-trivial.

The choice $h = \frac{1}{2}(p^2 + q^2)$, which fits our requirements, will not be analyzed here since it has been considered in all the details by Kaiser in his work, [1].

Example 4: $h = pq + \frac{i}{2}\mathbb{1}$

This is quite an easy application/example of our procedure, since the commutators between q, p and h are very easily handled. In fact, we will only need the BCH formula in its well known form $e^x e^y = e^{x+y+\frac{1}{2}[x,y]}$. It is possible to check that h is essentially self-adjoint in $\mathcal{L}^2(\boldsymbol{R})$, so that its extension is self-adjoint and, therefore, $W(\theta) = e^{i\theta h}$ is an unitary operator. Since $q(\theta)=qe^\theta$ and $p(\theta)=pe^{-\theta}$, it is easy to obtain that $\omega(\theta)=\omega e^\theta$, $s(\theta)=se^{-\theta}$, $T=1\!\!1$ and $\Phi=1$. Therefore $\mathcal{F}_{1\!\!1,\theta}$ shares with $\mathcal{F}$ the nature of $(A,B)-$frame.

Example 5: $h=p^2+pq+\frac{i}{2}1\!\!1$

It is possible to show that h is unitarily equivalent to a self-adjoint operator, so that, again, $W(\theta)$ is unitary. We have $q(\theta)=qe^\theta+p(e^\theta-e^{-\theta})$ and $p(\theta)=pe^{-\theta}$. Using the BCH formula we get $e^{i\omega q(\theta)}=e^{i\omega e^\theta q}e^{i\omega(e^\theta-e^{-\theta})p}e^{\frac{-i\omega^2}{2}(1-e^{2\theta})}$. Therefore, the assumptions of the Criterion are satisfied by taking $\omega(\theta)=\omega e^\theta$, $s(\theta)=se^{-\theta}-\omega(e^\theta-e^{-\theta})$, $T=1\!\!1$ and $\Phi=e^{\frac{i\omega^2}{2}(1-e^{2\theta})}$. Again, if $\mathcal{F}$ is an $(A,B)-$frame, also $\mathcal{F}_{1\!\!1,\theta}$ is an $(A,B)-$frame.

Remarks:–(1) The first remark concerns the form of the different $q(\theta)$ and $p(\theta)$ in the examples above. It is evident that, in all these examples, we have obtained expressions of the kind $q(\theta)=q\alpha(\theta)+\pi(\theta,p)$ and $p(\theta)=p\beta(\theta)+\chi(\theta,q)$, which are exactly the ones already discussed in this Section. Therefore, the examples are concrete realizations of our quite general statement.

(2) Some of the operators h used above are non positive or, even worse, they have no lower bound. This is not dramatic in our strategy, since we do not care (and we do not need to care!) about the physical meaning of the h's. We only need their (essential) self-adjointness.

At this point we want to show what some of these examples explicitly become when the original frame is fixed. In particular, we will analyze the set of the following coherent functions:

$$\varphi_{\omega s}(x)=U(\omega)V(s)\varphi_o(x)=e^{i\omega x}\varphi_o(x-s)=\frac{1}{\pi^{1/4}}e^{i\omega x}e^{-(x-s)^2/2}.$$

It is well known that, for any square integrable function $f(x)$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega\,ds|<\varphi_{\omega s},f>|^2=2\pi\|f\|^2, \tag{3.8}$$

which shows, following the terminology introduced in references [9, 10], that the set $\{\varphi_{\omega s}:(\omega,s)\in\boldsymbol{R}^2\}$ is a 2π-tight continuous frame.

We can modify this frame following the path suggested in this article. Let us consider first the easiest application, which follows from example 2 of this Section. In this case we have $T=e^{3is\theta q^2}$ and $W=e^{i\theta q^3}$. Therefore it is evident that

$$\varphi_{\omega s}^{T\theta}(x)=e^{i\omega x}\varphi^{T\theta}(x-s)=\frac{1}{\pi^{1/4}}e^{i\omega x}e^{i\theta(x-s)^2(x+2s)}e^{-(x-s)^2/2}.$$

We can check that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega\,ds|<\varphi_{\omega s}^{T\theta},f>|^2=2\pi\|f\|^2,\quad\forall f\in\mathcal{L}^2(\boldsymbol{R}), \tag{3.9}$$

which is not surprising also because $\varphi_{\omega s}^{T\theta}(x)$ differs from $\varphi_{\omega s}(x)$ only for a x-dependent phase factor. Equation (3.9) in particular shows that the set $\{\varphi_{\omega s}^{T\theta} : (\omega s) \in \boldsymbol{R}^2\}$ is still a 2π-tight continuous frame, in agreement with our Criterion.

We repeat now these same steps for the Example 1 above. In this case we have $T = e^{3i\omega\theta p^2}$ and $W = e^{i\theta p^3}$. After some computations we get the following integral expression for $\varphi_{\omega s}^{T\theta}$:

$$\varphi_{\omega s}^{T\theta}(x) = \frac{e^{i\omega x}}{\sqrt{2\pi}\pi^{1/4}} \int_{-\infty}^{\infty} dp' e^{-ip'(x-s)+3i\theta\omega p'^2+i\theta p'^3} e^{-p'^2/2}.$$

Again, it is only a matter of (easy) computations to show that the set $\{\varphi_{\omega s}^{T\theta} : (\omega, s) \in \boldsymbol{R}^2\}$ is a 2π-tight continuous frame.

We conclude this note stating our plans for the future.

First of all, a comment is in order: in our opinion, what is still missing in the literature, is some explicit physical applications in which the presence of these families of vectors in $\mathcal{L}^2(\boldsymbol{R})$ is really suggested by the problem itself. Nowadays, the utility of coherent states, wavelets and all these frame-related objects is out of doubt. This is what encourages us to believe that the freedom naturally contained in our approach (different h's give different frames!) will be useful in the analysis of some physical relevant problem.

From a mathematical point of view, many problems can still be discussed: it is possible to find a class of self-adjoint operators, different from the one already discussed in this Section (polynomials in q and p), such that $q(\theta)$ and $p(\theta)$ have the convenient form discussed here? Or, more generally, is it possible to obtain an a-priori way to check if a given self-adjoint operator satisfy or not the Criterion? And is it possible to relate directly $\omega(\theta)$, $s(\theta)$, Φ and T to h? Finally, it may be of a certain interest to discuss what happens if we substitute Gabor frames with wavelet frames, extending, in a sense, the analysis proposed in reference [2].

We hope to be able to give some answers to these problems in a near future.

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References

[1] G. Kaiser, *Deformations of Gabor Frames*, J. Math. Phys., **35**, 1372-1376 (1994)

[2] R.G. Baraniuk, D.L. Jones, Shear Madness: New Orthonormal Bases and Frames Using Chirp Functions, IEEE Trans. Sign. Proces. **41**, 3543-3548 (1993)

[3] G. Kaiser, *A Friendly Guide to Wavelets*, Birhäuser, Boston (1994)

[4] F. Bagarello, *Some Results about Frames*, N. Cimento B, **112**, N. 7, 961-980, (1997)

[5] I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, Philadelphia (1992)

[6] J.R. Klauder, B.S. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics*, World Scientific, Singapore, (1985)

[7] A.M. Perelomov, *On the Completeness of a System of Coherent States*, Theor. Math. Phys., **6**, 154-164, (1971)

[8] V. Bargmann, P. Butera, L. Girardello, J.R. Klauder, *On the Completeness of the Coherent States*, Rep. Math. Phys., **2**, 221-228, (1971)

[9] S.T. Ali, J.-P. Antoine, J.-P. Gazeau, *Square integrability of group representations on homogeneous spaces. II. Coherent and quasi-coherent states. The case of the Poincaré group*, Ann. Inst. H. Poincaré, **55**, n. 4, 857-890 (1991)

[10] S.T. Ali, J.-P. Antoine, J.-P. Gazeau, *Continuous frames in Hilbert Spaces*, Ann. Phys., **222**, 1-37 (1993)

[11] G.H. Weiss, A.A. Maradudin, *The Baker-Hausdorff Formula and a Problem in Crystal Physics*, J. Math. Phys., **3**, 771-777 (1962)

[12] W. Magnus, *On the Exponential Solution of Differential Equations for a Linear Operator*, Commun. Pure Appl. Math., **VII**, 649-673 (1954)

@inproceedings{Kaushik2014ANO, title={A note on Gabor frames}, author={S. K. Kaushik and Suman Panwar}, year={2014} }. S. K. Kaushik, Suman Panwar. Wilson frames $\{\psi_j^k :w_0,w_{-1}\in L^2(\mathbb{R})\}_{j\in\mathbb{Z}\atop {k \in\mathbb{N}_0}}$ in $L^2(\mathbb{R})$ have been defined and a characterization of Wilson frames in terms of Gabor frames is given when $w_0=w_{-1}$. Also, under certain conditions a necessary condition. for a Wilson system to be a Wilson Bessel sequence is given. We have also obtained sufficient conditions for a Wilson system to be a Wilson frame in terms of Gabor (vi) Note that the 2d-dimensional Gaussian is the tensor product of two d-dimensional. Gaussian. Writing Ï‰ = (Ï‰1, Ï‰2), x = (x1, x2), we also have MÏ‰Txg = MÏ‰1 Tx1 g âŠ— MÏ‰2 Tx2 g and thusÂ The next theorem nally. establishes the existence of Gabor frames.Â As was shown in the rst chapter Gabor frames provide a method of analysing and synthesising functions like for instance audio signals. So letâ€™s have a look at one possible application - wireless communication. The interesting part of the signal are. 2. Nonstationary Gabor Frames. Frames were rst mentioned in [9], also see [5, 6]. Frames are a generalization of (orthonormal) bases and allow for redundancy and thus for much more exibility in design of the signal representation. Thus, frames may be tailored to a specic appli-cation or certain requirements such as a constant-Q frequency resolution. Loosely speaking, we wish to expand, or represent, a given signal of interest as a linear combination of some building blocks or atoms Ï•n,k, with (n, k) âˆˆ Z Ã— Z, which are the members of our frame: (1) f = cn,kÏ•n,k. n,k.Â for all f âˆˆ H. Note that, if the set of functions {Ï•n,k, (n, k) âˆˆ Z Ã— Z} is an orthonormal basis, then S is the identity operator. If S is invertible on H, then.