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A Material that Ought to find its Place in Future Strength of Materials Textbooks*

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A material is exposed that appears to be appropriate for the future strength of materials textbooks. The described solution is simpler than the classical solution by Euler derived over 250 years ago. The material, although elementary in mathematical terms, represents a simple example of a semi-inverse design problem and leads to the closed-form solution. The material can be covered within 2 lecture hours of 50 minutes duration each.

INTRODUCTION

ABOUT 250 YEARS AGO, Leonhard Euler solved the first buckling problems [1, 2]. Namely, in 1744 he first solved a nonlinear buckling problem whereas in the later publication he tackled the column's buckling under a concentrated compressive loads, in the linear setting. (For the exposition of Euler's original papers one can consult with the paper by van der Broek [3].) The solution for the buckling load and the associated buckling mode for the uniform column that is simply supported at both ends is an invariable element for any textbook on the strength of materials and the mechanics of solids. Let us recapitulate this solution. The governing differential equation reads:

$$EI\frac{d^2y}{dx^2} + P_{cr}y = 0 \tag{1}$$

where E = modulus of elasticity, I = moment of inertia of the cross section, $y(x)$ = buckling mode, x = axial coordinate, P = axial compression whose critical value is being sought. The boundary conditions read:

$$y = 0 \quad at \quad x = 0, x = L \tag{2}$$

where L = length of the column. Since EI = const we divide both sides of Equation (1) by it, and get:

$$y'' + k^2y = 0 \tag{3}$$

where

$$k^2 = P_{cr}/EI \tag{4}$$

Equation (3) represents the ordinary differential equation with a constant coefficient. Its solution reads:

$$y = D_1 \sin kx + D_2 \cos kx \tag{5}$$

Due to the condition $y(0) = 0$, we get $D_2 = 0$, and are left with:

$$y = D_1 \sin kx \tag{6}$$

in the expression for the buckling mode.

Satisfying the condition $y(L) = 0$, we obtain:

$$D_1 \sin kL = 0 \tag{7}$$

Here we have two possibilities. The first one is associated with the requirement $D_1 = 0$. This would mean, by virtue of Equation (6), that $w(x) \equiv 0$, i.e. the column remains straight throughout its deformation. This contradicts the condition that $y(x)$ does not vanish automatically, for we are looking for the buckling situation. Thus:

$$D_1 \neq 0 \tag{8}$$

We are left with the condition:

$$\sin kL = 0 \tag{9}$$

which yields:

$$kL = \pi \tag{10}$$

as a first nontrivial solution for non-zero k. Thus, recalling the definition (Equation 4) of k, we get the value of P_{cr} that corresponds to the displacement $y(x)$ that is not zero everywhere:

$$P_{cr} = \pi^2 EI/L^2 \tag{11}$$

This is a famous formula of Euler, and is uniformly known to the students and engineers, over the past two centuries. Its active use was witnessed in the past century with the advent of the technological revolution that is demanded for design of lightweight structures.

A natural question arises: Is there any *simpler* problem on buckling of elastic columns? At the first glance one cannot imagine a simpler buckling problem. Still, it turned out that this inquiry is not trivial. According to Einstein, a mere formulation

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often may constitute a more important step than the solution. Here, we present a different formulation and solution of the buckling problem. It constitutes a semi-inverse problem, namely, the buckling mode is postulated, and an inhomogeneous column is constructed that possesses the pre-selected buckling mode.

SEMI-INVERSE BUCKLING PROBLEM

The above problem associated with uniform Euler column can be classified as a *direct* one, implying that the flexural rigidity:

$$D = EI \tag{12}$$

is known and one has to find the critical value P_{cr} of the load as well as the associated buckling mode $y(x)$. In most cases the determination of the buckling mode is a difficult task and may lead to solution in trascendental functions like Bessel functions [4] and Lommel functions [5].

Elishakoff [6] posed and solved several semi-inverse problems. In our context such a problem reads: Construct a nonhomogenoeus column, with variable flexural rigidity:

$$D(x) = E(x)I(x), \tag{13}$$

so that the buckling mode is a preselected function $f(x)$, i.e.

$$y(x) = f(x) \tag{14}$$

The governing differential equation for the non-uniform column reads:

$$D(x)f'' + P_{cr}f = 0 \tag{15}$$

since we postulated the knowledge of the buckling mode $f(x)$ in Equation (14), we get directly the buckling load from Equation (15):

$$P_{cr} = -D(x)f''(x)/f \tag{16}$$

The natural question arises: Which function $f(x)$ must be preselected? Naturally it is the best to look for simplest possible candidate functions $f(x)$. Note that Equation (16) is also a focal point of the 'method of assuming the exact solution' by Zyczkowski [7, 8].

CHOICE OF THE POSSIBLE BUCKLING MODE

The simplest function that satisfies the boundary conditions in Equation (2) is a second-order polynomial. We express it as follows:

$$f(x) = a_0 + a_1x + a_2x^2 \tag{17}$$

Satisfaction of the boundary condition $f(0) = 0$ yields $a_0 = 0$. At $x = L$, we must have $f(L) = 0$, i.e.

$$a_1 + a_2L = 0 \tag{18}$$

or,

$$a_1 = -a_2L \tag{19}$$

Thus,

$$f(x) = -a_2(xL - x^2) \tag{20}$$

SOLUTION OF THE SEMI-INVERSE PROBLEM

The second term $P_{cr}f$ in Equation (15), once $f(x)$ is substituted for $f(x)$, is a second-order polynomial:

$$P_{cr}f = a_2(P_{cr}xL - P_{cr}x^2) \tag{21}$$

In order for the polynomial mode shape to be allowable, the first term $D(x)y' = D(x)f'$ in Equation (15) ought to be also a second-order polynomial. Since f is a constant, we conclude that $D(x)$ ought to be a second-order polynomial. We express it as follows:

$$D(x) = b_0 + b_1x + b_2x^2 \tag{22}$$

and get

$$\begin{aligned} D(x)f'' &= -a_2(b_0 + b_1x + b_2x^2)(-2) \\ &= -a_2(-2b_0 - 2b_1x - 2b_2x^2) \end{aligned}$$

Since Equation (15) must hold, we obtain, by substituting Equation (21) and (22) into it:

$$-a_2(-2b_0 - 2b_1x - 2b_2x^2 + P_{cr}xL + P_{cr}x^2) = 0 \tag{24}$$

Since this equation must hold for any x, we get:

$$\begin{aligned} 2b_0 &= 0 \\ -2b_1 + P_{cr}L &= 0 \\ -2b_2 - P_{cr} &= 0 \end{aligned} \tag{25}$$

These equations result in:

$$\begin{aligned} b_0 &= 0 \\ P_{cr} &= -2b_1/L \\ P_{cr} &= -2b_2 \end{aligned} \tag{26}$$

For these equations to be compatible, we stipulate:

$$b_1 = -b_2L \tag{27}$$

In order that equations $P_{cr} = 2b_1/L$ and $P_{cr} = -2b_2$ express the buckling load, b_2 must be negative. For the flexural rigidity we get:

$$D(x) = -b_2Lx + b_2x^2 = -b_2(Lx - x^2) \tag{28}$$

Since b_2 is negative, we arrive at a conclusion that the flexural rigidity is a positive valued function except points $x = 0$ and $x = L$ where it vanishes. The results may be put in the following form:

$$P_{cr} = 2|b_2| \tag{29}$$

$$D(x) = |b_2|\left(xL - x^2\right) \tag{30}$$

As is seen, we obtained a solution within parameter $|b_2|$ that is arbitrary positive constant. This implies that the posed problem has an infinite number of solutions. Specific solutions can be obtained once one preselects the value of b_2. For example, if $b_2 = -1$, we get $P_{cr} = 2$ and $D(x) = xLx^2$; likewise, if b^2 is taken equal -5, the critical value of the load is $P_{cr} = 10$, while the appropriate flexural rigidity equals $D(x) = 10(xLx^2)$ and so on.

DESIGN PROBLEM

This above solution permits us to solve some design problems. The basic design problem is given as follows. Construct a column that has a buckling load that is the preselected load value $P_{desired}$, Equation (29) yields then:

$$2|b_2| = P_{desired} \tag{31}$$

Resulting in:

$$|b_2| = P_{desired}/2 \tag{32}$$

The appropriate flexural rigidity is obtained by substituting Equation (32) into Equation (30):

$$D(x) = \tfrac{1}{2}P_{desired}(xL - x^2) \tag{33}$$

COMPARISON WITH THE GALERKIN METHOD

Since an exact solution has been derived, it can be used as a benchmark problem. Then the accuracy of approximate methods can be checked by contrasting them with the above closed-form solution of the buckling load.

As approximate methods one can use, for example, the Rayleigh-Ritz method, Boobnov-Galerkin method, Timoshenko method, finite-difference, finite element or differential quadrature methods. Here we choose the Boobnov-Galerkin method. Using a single-term approximation with the comparison function $\sin(\pi x/L)$ yields the approximation:

$$P_{cr,app} = \frac{(\pi^2 + 3)}{6}|b_2| \tag{34}$$

or $P_{cr,app} = 2.1449|b_2|$ constituting a 7.2% error. Two term-approximation with the trial function $\sin(\pi x/L)$ and $\sin(3\pi x/L)$ yields the following expression:

$$P_{cr} = 2.0458|b_2| \tag{35}$$

with attendant 2.3% error.

DUNCAN'S EXAMPLE AS A PARTICULAR CASE

To construct a candidate mode shape, consider a uniform beam with flexural rigidity $(EI)_0$ under linearly distributed load αx. The governing equation:

$$(EI)_0\frac{d^4w}{dx^4} = \alpha x \tag{36}$$

has a straightforward solution:

$$w(x) = \frac{\alpha}{360(EI)_0}(7xL^4 - 10x^3L^2 + 3x^5) \tag{37}$$

One can pose the following question: Is there an inhomogeneous column whose buckling mode is proportional to the expression in parentheses?

$$y(x) = \gamma\left(7xL^4 - 10x^3L^2 + 3x^5\right) \tag{38}$$

To reply to this question we substitute this expression into Equation (16) to get:

$$P_{cr} = \frac{D(x)60(x^3 - xL^2)}{3x^5 - 10x^3L^2 + 7L^4x} \tag{39}$$

Now if $D(x)$ is proportional to the ratio of $3x^5 - 10x^3L^2 + 7xL^4$ to $x^3 - xL^2$, which is $3x^2 - 7L^2$, such an inhomogeneous column exists. Now if:

$$D(x) = a^2\left[1 - \frac{3}{7}\left(\frac{x}{L}\right)^2\right] \tag{40}$$

Where $a^2 = D(0)$, then the buckling load is obtained as:

$$P_{cr} = 60a^2/7 \tag{41}$$

which coincides with Duncan's example [9]. Duncan *guessed* the buckling mode, and appropriate distribution of the flexural rigidity (42), whereas here the expressions have been derived by a general procedure.

DISCUSSION

It is important to note that Equation 16 can also be utilized to correlate with the obtained solution in equations (29) and (30). Indeed, once the buckling mode shape in Equation 20 is established, it can be substituted into Equation 16. This leads to:

$$P_{cr} = \frac{2D(x)}{xL - x^2} \tag{42}$$

since $f'' = 2a_2$. We conclude that in order P_{cr} to be a constant, the flexural rigidity $D(x)$ must be proportional to the denominator:

$$D(x) = a^2(xL - x^2) \tag{43}$$

where a^2 is the coefficient of proportionality. It must be positive due to non-negativity requirement imposed upon the flexural rigidity. Thus:

$$P_{cr} = 2a^2 \tag{44}$$

Comparison with Equation 29 reveals that $|b_2| = a^2$.

The solution derived in Equation (29) and (30) is simpler than Equation (11) due to the Euler's method. Moreover, Euler's formula includes an irrational number π, whereas the present solution in formulated in terms of rational expressions. Scientists know π with the accuracy of over one billion digits, and the critical load in Equation 11 can be evaluated with arbitrary accuracy. Still, a rational expression $P_{cr} = 2|b_2|$ appears to be superior.

One can pose the following question. In the derived solution the flexural rigidity vanishes at the column's ends. Can one get a column, via the above semi-inverse method, that does not have a vanishing flexural rigidity? The reply is affirmative, if one employs *other* postulated buckling modes. For example, if one uses the following mode:

$$f(x) = xL^3 - 2x^3L + x^4 \tag{45}$$

we get, by substituting into Equation 16:

$$P_{cr} = -\frac{12D(x)(x^2 - xL)}{x^4 - 2x^3L + xL^3} \tag{46}$$

or

$$P_{cr} = \frac{12D(x)}{L^2 + xL - x^2} \tag{47}$$

Thus, if *D(x)* takes is proportional to the denominator:

$$D(x) = a^2(L^2 + xL - x^2) \tag{48}$$

then

$$P_{cr} = 12a^2 \tag{49}$$

Moreover, $D(x)$ takes on positive values throughout the interval $[0, L]$. In this case in order for the column to buckle at the value $P_{desired}$, the flexural rigidity must be chosen as:

$$D(x) = \frac{P_{desired}}{12}(L^2 + xL - x^2) \tag{50}$$

The expressions (49) and (50) were obtained by Elishakoff [6] by alternative means, by using a fourth-order governing differential equation for the column at buckling. The use of the second-order differential equation, adopted in this paper, is simpler.

Note that the expression (45) is proportional to the static deflection of the uniform beam under distributed load. This leads to a remarkable conclusion: the static deflection of the uniform column may serve as the buckling mode of the inhomogeneous column.

In Elishakoff's study [10], another intriguing fact was uncovered. Not only the *static* deflection of the associated uniform beam, but also its *vibration* mode of a *uniform* column may be utilized as the postulated buckling mode of an *inhomogeneous* column. Thus, the behavior of inhomogeneous columns is (naturally) much richer than that of the inhomogeneous ones.

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APPENDIX

We look for the fourth order polynomial:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \tag{A-1}$$

as a candidate function for the buckling mode to be postulated. The requirement $f(0) = 0$, leads to $a_0 = 0$. The condition $f'(0) = 0$ yields $a_2 = 0$. Thus we have:

$$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \tag{A-2}$$

The conditions $f(L) = 0, f'(L) = 0$ result in:

$$a_1 L + a_3 L^3 + a_4 L^4 = 0 \tag{A-3}$$

$$6a_3 L^3 + 12 a_4 L^4 = 0 \tag{A-4}$$

From Equation (A-4):

$$a_3 = -2a_4 L \tag{A-5}$$

From Equation (A-3), in view of (A-5), we get:

$$a_1 = a_4 L^3 \tag{A-6}$$

Leading to:

$$f(x) = a_4\left(xL^3 - 2x^3 L + x^4\right) \tag{A-7}$$

We fix $a_4 = 1$, since it is an arbitrary constant. Thus the postulated buckling mode reads:

$$f(x) = xL^3 - 2x^3 L + x^4 \tag{A-8}$$

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