AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **58(2)** (2014), Pages 249–263

A construction of a class of graphs with depression three*

C. M. Mynhardt† M. Schurch

Department of Mathematics and Statistics
University of Victoria, P.O. Box 3060 STN CSC
Victoria, BC V8W 3R4
Canada
mynhardt@math.uvic.ca mschurch@uvic.ca

Abstract

An *edge ordering* of a graph G is an injection $f : E \to \mathbb{R}$, the set of real numbers. A path in G for which the edge ordering f increases along its edge sequence is called an f-*ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *depression* of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k. We provide a construction of a large class of graphs with depression three.

1 Introduction

An *edge ordering* of a graph G is an injection $f : E(G) \to \mathbb{R}$, the set of real numbers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path λ in G for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an f-*ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *flatness* of an edge ordering f, denoted by $h(f)$, is the length of a shortest maximal f-ascent of G. In [9] it was shown that for a given edge-ordering f of a graph G the problem of determining the value of $h(f)$ is NP-hard.

The *depression* of G was defined in [6] as $\varepsilon(G) = \max_{f\in\mathcal{F}(G)} \{h(f)\}$. The interpretation of the depression of a graph G is that any edge ordering f has a maximal f-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true.

Clearly, $\varepsilon(G) = 1$ if and only if K_2 is a component of G. Graphs with depression two were characterized in [6], while trees with depression three were characterized

---

* This paper is based on part of Mark Schurch's doctoral dissertation.

† Supported by an NSERC discovery grant.

in [10]. Graphs with depression three and no adjacent vertices of degree three or higher were characterized in [13]. In this paper we further investigate graphs with depression three and describe a construction of a large class of graphs with depression three, which includes cyclic graphs and graphs with adjacent vertices of high degree. This paper is based on part of the second author's dissertation [15].

2 Definitions and Background

We consider simple, finite graphs $G = (V(G), E(G))$. For basic graph theoretic definitions we refer the reader to the book [4] or any of its predecessors. The *open neighbourhood* of a vertex v of G is the set of all vertices adjacent to v and is denoted by $N_G(v)$, or just $N(v)$, and its *closed neighbourhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$.

Consider two disjoint graphs G_1 and G_2 and vertices $v_i \in V(G_i)$. The *vertex-coalescence of* G_1 *and* G_2 *via* v_1 *and* v_2 is the graph obtained by identifying v_1 and v_2 to form a new vertex v, and is denoted $(G_1 \cdot G_2)(v_1, v_2 : v)$. In forming $G = (G_1 \cdot G_2)(v_1, v_2 : v)$, if v_2 is unimportant we also say we *attach* G_1 *to* G_2 *at* v_1, and if G is the resulting graph, we say that G contains G_1 *as an attachment at* v_1.

A *branch vertex* of a tree is a vertex with degree at least three. Let $B(T)$ and $L(T)$ respectively denote the sets of all branch vertices and all leaves of the tree T. For $v \in V(T)$ and $l \in L(T)$, a (v, l)-*endpath*, or v-*endpath* if l is unimportant, or *endpath* if neither v nor l is important, is a path P from v to l such that each internal vertex of P has degree two in T. A *spider* $S(a_1, a_2, ..., a_r)$ is a tree with exactly one branch vertex v and v-endpaths of lengths $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$, where $r = \deg v$.

Given an edge ordering f of the graph G, an f-ascent λ is simply called an *ascent* if the ordering is clear, and if λ has length k, it is also called a (k, f)-*ascent*. If the path λ with vertex sequence $v_0, v_1, ..., v_k$ or edge sequence $e_1, e_2, ..., e_k$ forms an f-ascent, we denote this fact by writing λ as $v_0v_1...v_k$ or $e_1e_2...e_k$. which $f \in \mathcal{F}(G)$ increases along the edges of P, is called a *u-v direct f-ascent*, or a *direct f-ascent* if u and v are clear, or simply a *direct ascent* if u, v, and f are clear.

We emphasize that to show that $\varepsilon(G) = k$, we must show that

(a) each edge ordering of G has a maximal ascent of length at most k – this shows that $\varepsilon(G) \leq k$,

(b) there exists an edge ordering f of G with no maximal ascents of length less than k, i.e. for which each (l, f)-ascent, where $l < k$, can be extended to a (k, f)-ascent – this shows that $\varepsilon(G) \geq k$.

The *height* of an edge ordering f, denoted $H(f)$, is the length of a longest f-ascent of G. In [2] the *altitude* of G was defined as $\alpha(G) = \min_{f \in \mathcal{F}(G)} \{H(f)\}$. The interpretation of the altitude of a graph G is that any edge ordering $f \in \mathcal{F}(G)$ has an f-ascent of length at least $\lambda(G)$, and $\lambda(G)$ is the largest integer for which this statement is true.

The study of lengths of increasing paths was initiated by Chvátal and Komlós [5] who posed the problem of determining the altitude of the complete graph. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [2, 5]). The altitude of graphs was also investigated in e.g. [1, 2, 3, 8, 9, 11, 14, 16].

3 Known Results

Let $\tau(G)$ denote the length of a longest path in G, called the *detour length* in G. If we assume that G is connected and of size at least two, then

$$2 \leq \varepsilon(G), \alpha(G) \leq \tau(G).$$

By taking the edge ordering f for the path P_n, $n \geq 3$, to increase along its edge sequence we see that $\varepsilon(P_n) = \tau(P_n) = n - 1$. On the other hand, by taking the edge ordering for the path P_n, $n \geq 3$, as $1, n-1, 2, n-2, ..., \lceil \frac{n}{2} \rceil$ along its edge sequence, we see that $\alpha(P_n) = 2$.

If a connected graph G has a vertex v that is adjacent to u, w, where u, w are end-vertices or adjacent vertices of degree two, then in any edge ordering f of G, either u, v, w or w, v, u is a maximal $(2, f)$-ascent, hence $\varepsilon(G) = 2$. In [6] it was shown that the converse of this statement is also true, which gives the following characterization of graphs with depression two.

Theorem 1. [6] *If G is connected, then $\varepsilon(G) = 2$ if and only if G has a vertex adjacent to two end-vertices or to two adjacent vertices of degree two.*

It is reasonable to expect a link between the depression of a graph and the diameter of its line graph, and indeed the following result appeared in [6].

Theorem 2. [6] *If* $\operatorname{diam} L(G) = 2$, *then* $\varepsilon(G) \leq 3$.

However, the difference $\operatorname{diam} L(G) - \varepsilon(G)$ can be arbitrarily large, a result that easily follows from Theorem 1. Much harder to see is that the difference $\varepsilon(G) - \operatorname{diam} L(G)$ can also be arbitrarily large, as shown by Gaber-Rosenblum and Roditty in [7].

We see from Theorem 1 that if v is the central vertex of P_3 or any vertex of K_3, and G is any connected graph containing P_3 or K_3 as an attachment at v, then $\varepsilon(G) = 2$.

An interesting question arises from this result.

- If H is a graph with $\varepsilon(H) = k$ and $v \in V(H)$, what properties should H and v satisfy so that if we attach an arbitrary graph to H at v, the resulting graph has depression at most k?

To help answer this question, a *k-kernel* of a graph G is defined in [10] as a set $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal (l, f)-ascent

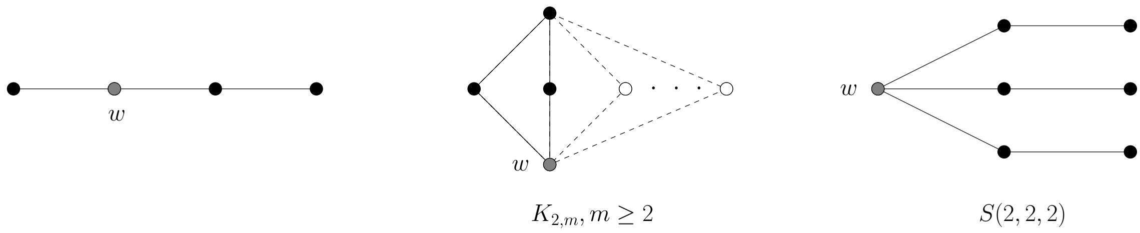

Figure 1: The set of graphs $\mathcal{H}$.

for some $l \leq k$ that neither starts nor ends at a vertex in U and k is the smallest value for which this is true. For example, it is easy to verify that any vertex of P_4 with degree two is a 3-kernel of P_4. If an f-ascent λ neither starts nor ends in a set $A \subset V(G)$, we say that λ is an A-*avoiding* (*maximal*) f-ascent or an a-*avoiding* (*maximal*) f-ascent if A contains a single vertex a (and λ is not contained in a longer f-ascent). The following theorem relates the concept of kernels to the question above.

Theorem 3. [10] *Let H be an arbitrary graph and let U be a k-kernel of H. Form a graph G by adding any set A of new vertices and arbitrary edges joining vertices in $U \cup A$. Then $\varepsilon(G) \leq k$.*

Therefore, if G has a non-empty k-kernel, Theorem 3 provides us with a method of forming a family of graphs with depression at most k. For example, if v is a vertex of P_4 with degree 2 and G is any graph that contains P_4 as an attachment at v, then by Theorem 3, $\varepsilon(G) \leq \varepsilon(P_4) = 3$.

The following theorem describes a necessary condition for a vertex v to be a k-kernel of a graph G with $\operatorname{diam}(L(G)) = 2$, where $k \in \{2, 3\}$.

Theorem 4. [12] *Let G be a graph with* $\operatorname{diam}(L(G)) = 2$. *If v is a vertex such that $N[v]$ is a vertex cover of G, then v is a k-kernel of G for some $k \in \{2, 3\}$.*

Theorem 4 allows one to construct a large class of graphs with depression three. For example, the line graph of any complete graph K_n with $n \geq 4$ has diameter two, and for any vertex $v \in K_n$, $N[v]$ is a vertex cover of K_n. Therefore, it follows from Theorem 4 that any graph G with an end-block $B \cong K_n$, where $n \geq 4$, has depression at most three.

Graphs with depression three and no adjacent vertices of degree three or more were characterized in [13].

Let $\mathcal{H}$ be the set of graphs consisting of P_4, $K_{2,m}$ for $m \geq 2$, and the spider $S(2, 2, 2)$– see Figure 1. For each graph in Figure 1 the vertex labelled w is a 3-kernel of its associated graph.

Theorem 5. [13] *Let G be a connected graph with* $\operatorname{diam}(L(G)) \geq 3$, *no vertex adjacent to two end-vertices or to two adjacent vertices of degree two, and no adjacent vertices of degree three or more. Then $\varepsilon(G) = 3$ if and only if $G = S(2, 2, 2)$, or for some $H \in \mathcal{H}$, G contains H as an attachment at a vertex which is a 3-kernel of H.*

The following characterization of trees with depression three was given in [10].

Let $\mathcal{S}_k$ be the class of trees S_k, $k \geq 1$, that can be constructed recursively as follows. Let $S_0 = K_2$ with $V(S_0) = \{\alpha, \alpha'\}$. Define $U_0 = \emptyset$ and $Y_0 = \{\alpha\}$. Once S_i has been constructed, construct S_{i+1} by performing one of the following two operations.

O1: For any $y \in Y_i$, join y to the vertex u of a new edge ux; let $U_{i+1} = U_i \cup \{u\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join y to the central vertex w of a new $P_5 : s, r, w, t, z$; let $U_{i+1} = U_i \cup \{w\}$ and $Y_{i+1} = Y_i \cup \{r, t\}$.

Let $\mathcal{S} = \bigcup_{k=1} \mathcal{S}_k$. Note that $S_0 = K_2$ is not in $\mathcal{S}$. For a tree $S \in \mathcal{S}$, define $U_S = U_k$. Let $\mathcal{G}$ be the class of all graphs G_S constructed as follows.

O3: Add any set $A = A(G_S)$ of new vertices to a tree $S \in \mathcal{S}$ and arbitrary edges between vertices in $A \cup U_S$.

Let $\mathcal{T} = \{T \in \mathcal{G} : T \text{ is a tree}\}$.

Theorem 6. [10] *For any tree T, $\varepsilon(T) = 3$ if and only if $T \in \mathcal{T}$ and no vertex of T is adjacent to two leaves.*

The main result of this paper is a generalization of this characterization of trees with depression three.

4 Main Result

In this section we provide a construction of a large class of graphs with depression three which includes acyclic graphs and graphs with adjacent vertices of high degree. The construction is a generalization of the construction used in [10] to characterize trees with depression three.

Let $\mathcal{S}'_k$ be the class of graphs S_k, $k \geq 1$, that can be constructed recursively in k steps as follows. Let $S_0 = K_2$ with $V(S_0) = \{x_0, y_0\}$. Define $U_0 = \emptyset$ and $Y_0 = \{y_0\}$. Once S_i has been constructed, construct S_{i+1} by performing one of the following five operations.

O1: For any $y \in Y_i$, join y to the vertex u_1 of a new edge u_1x_1; let $U_{i+1} = U_i \cup \{u_1\}$ and $Y_{i+1} = Y_i$.

O2: For any $y \in Y_i$, join y to the central vertex u_2 of a new $P_5 : x_2, y_2, u_2, y'_2, x'_2$; let $U_{i+1} = U_i \cup \{u_2\}$ and $Y_{i+1} = Y_i \cup \{y_2, y'_2\}$.

O3: For any $y \in Y_i$, join y to the vertices u_3 and v_3 of a new edge u_3v_3; let $U_{i+1} = U_i \cup \{u_3\}$ and $Y_{i+1} = Y_i$.

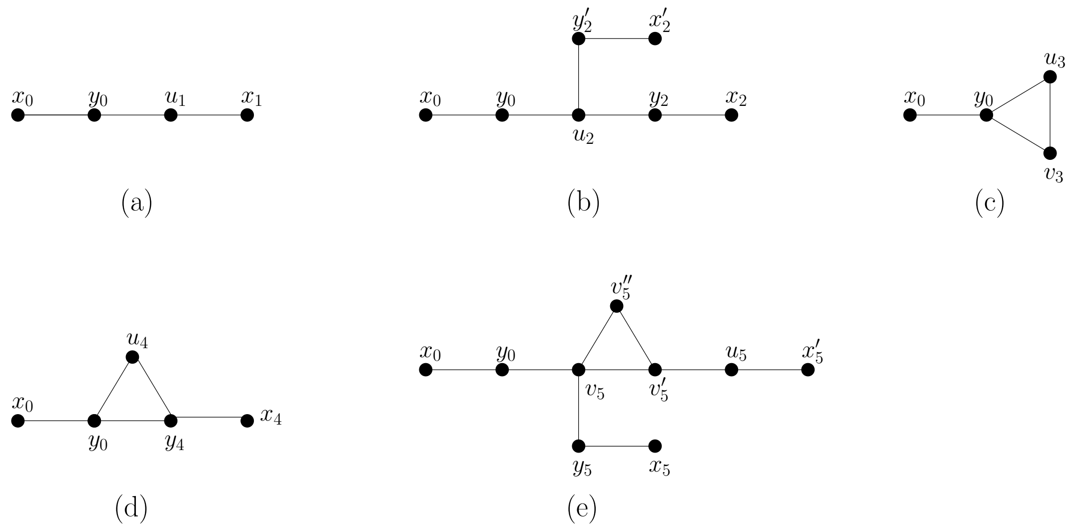

Figure 2: S_1 for each of the five operations **O1-O5**.

O4: For any $y \in Y_i$, join y to the central vertex y_4 and an end vertex u_4 of a new $P_3 : u_4, y_4, x_4$; let $U_{i+1} = U_i \cup \{u_4\}$ and $Y_{i+1} = Y_i$.

O5: For any $y \in Y_i$, join y to the vertex v_5 of the graph $G_5 = (\{x_5, x_5', v_5, v_5', v_5'', u_5, y_5\}, \{v_5y_5, y_5x_5, v_5v_5', v_5v_5'', v_5'v_5'', v_5'u_5, u_5x_5'\})$; let $U_{i+1} = U_i \cup \{u_5\}$ and $Y_{i+1} = Y_i \cup \{y_5\}$.

The operations **O1-O5** performed on S_0 are illustrated in Figure 2.

Let $\mathcal{S}_k$ be the family of graphs such that $S_k \in \mathcal{S}_k$ whenever $S_k \in \mathcal{S}_k'$ and in the construction of S_k, any vertex $y \in Y_k$ is involved in **O3** at most once. Define $\mathcal{S} = \bigcup_{k \geq 1} \mathcal{S}_k$. Note that $S_0 = K_2$ is not in $\mathcal{S}$. For a graph $S = S_k \in \mathcal{S}$, define $U_S = U_k$ and $Y_S = Y_k$. Let $\mathcal{G}$ be the class of all graphs G_S formed by performing the following two operations.

O6: Add any set $A = A(G_S)$ of new vertices to a graph $S \in \mathcal{S}$ and arbitrary edges between vertices in $A \cup U_S$.

O7: Add any arbitrary edges between vertices in Y_S.

Remark 7. *Let $S \in \mathcal{S}$. The operations **O1-O5** show that if $y \in Y_S$, then y is adjacent to exactly one vertex of degree one.*

We define the following property for a graph G.

P1: A graph G has property **P1** with respect to an edge ordering f and sets $U_G, Y_G \subseteq V(G)$, if for each $y \in Y_G$ for which a U_G-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends (starts) at y, there exists a U_G-avoiding maximal $(2, f)$- or $(3, f)$-ascent for which its last (first) edge is assigned the largest (smallest) value under f over all edges incident with y.

Lemma 8. *If $S \in \mathcal{S}$ and f is an edge ordering of S for which there exists a U_S-avoiding maximal f-ascent of length at most three and all such ascents start or end in Y_S, then S has property **P1** with respect to f, U_S and Y_S.*

Proof. Let $y \in Y_S$ be a vertex for which a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent ends at y, A_y be the set of all such f-ascents, and $\lambda = aby$ or $\lambda = acby$, where λ is the maximal f-ascent such that its last edge by is assigned the largest value over all edges of ascents in A_y. Let x be the end vertex adjacent to y. Clearly, $f(by) > f(yx)$.

Suppose to the contrary that $f(by) \neq \max_{v\in N(y)}\{f(vy)\}$. Then there exists an edge $wy \in E(S)$ such that $w \neq b$ and $f(wy) = \max_{v\in N(y)}\{f(vy)\}$. Since λ is a maximal f-ascent, w is a vertex of λ. By the construction of graphs in $\mathcal{S}$, all cycles of S have length three and we may assume that wby is a 3-cycle. If the cycle was introduced by **O3**, then $\lambda = wby$, $b \in U_S$, $w \notin U_S \cup Y_S$, and both w and b have degree 2. But since $f(yw) > f(wb)$ and $\deg(w) = 2$, xyw is a $U_S \cup Y_S$-avoiding maximal f-ascent, a contradiction.

Suppose then that the cycle wby was introduced by **O4**. Then $w \in Y_S$ and there exists an end vertex x' adjacent to w. If $f(x'w) < f(wy)$, then $x'wy$ is a maximal f-ascent, which contradicts our choice of λ. Now if $f(x'w) > f(wy)$, then $xywx'$ is a maximal f-ascent which is also a contradiction.

A similar argument may be used to show that if a U_S-avoiding maximal f-ascent of length at most three starts at y, then there exists a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent λ such that for the initial edge yb of λ, $f(yb) = \min_{v\in N(y)}\{f(yv)\}$. □

Theorem 9. *For each $S \in \mathcal{S}$, $\varepsilon(S) \leq 3$ and U_S is a k-kernel of S for some $k \in \{2, 3\}$.*

Proof. The proof is by induction on k, the number of steps used to construct $S = S_k$ from $K_2 = S_0$. To prove the result we must show that for any edge ordering f of S there exists a U_S-avoiding maximal $(2, f)$- or $(3, f)$-ascent.

If $k = 1$, then S was constructed by performing one of the operations **O1**-**O5** on $K_2 = S_0$

Case 1 **O1** is performed. Then $S = P_4$ and $U_S = \{u_1\}$. Since $\mathrm{diam}(L(S)) = 2$ and $N[u_1]$ is a vertex cover of S, the result follows from Theorem 4.

Case 2 **O2** is performed. Then $S = S(2, 2, 2)$ and $U_S = \{u_2\}$. Consider any edge ordering f of S. Without loss of generality we may assume $f(x_0y_0) < f(y_0u_2)$. If $f(y_0u_2) > y(u_2y_2)$, then either $x_2y_2u_2y_0$ (if $f(x_2y_2) < f(y_2u_2)$) or $y_2u_2y_0$ (if $f(x_2y_2) > f(y_2u_2)$) are u_2-avoiding maximal f-ascents of S with length at most three. The same argument applies if $f(y_0u_2) > f(u_2'y_2')$. Suppose then that $f(y_0u_2) < f(u_2y_2)$ and $f(y_0u_2) < f(u_2'y_2')$. To avoid a u_2-avoiding maximal f-ascents of length at most three, both $x_0y_0u_2x_2y_2$ and $x_0y_0u_2x_2'y_2'$ are maximal $(4, f)$-ascents of S. This implies either $y_2u_2y_2'x_2'$ (if $f(y_2u_2) < f(u_2y_2')$) or $y_2'u_2y_2x_2$ (if $f(y_2u_2) > f(u_2y_2')$) is a u_2-avoiding maximal f-ascent of the required length.

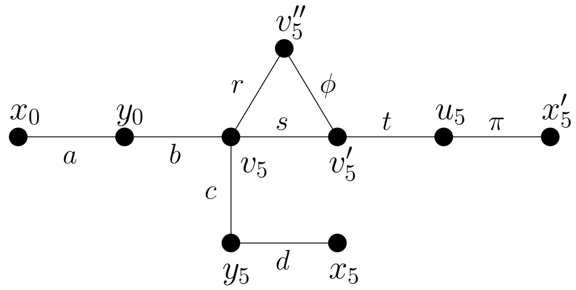

Figure 3: Operation O5 is performed, and the paths $abcd$ and rst are f-ascents of S.

Case 3 O3 is performed. Then $U_S = \{u_3\}$. Since $\mathrm{diam}(L(S)) = 2$ and $N[u_3]$ is a vertex cover of S, the result follows from Theorem 4.

Case 4 O4 is performed. Then $U_S = \{u_4\}$. Since $\mathrm{diam}(L(S)) = 2$ and $N[u_4]$ is a vertex cover of S, once again, the result follows from Theorem 4.

Case 5 O5 is performed. Then $U_S = \{u_5\}$. Suppose to contrary that u_5 is not a 3-kernel of S. Let f be an edge ordering f of S for which all maximal $(2, f)$- and $(3, f)$-ascents either start or end at u_5. Necessarily, either $x_0y_0v_5y_5x_5$ or its reverse is a $(4, f)$-ascent of S, and without loss of generality we assume the former. Furthermore, by our assumption, neither $v_5''v_5v_5'$ nor its reverse is a maximal $(2, f)$-ascent of S, which implies either $v_5''v_5v_5'u_5$, $v_5''v_5v_5'u_5x_5'$, or the reverse of one of these paths is a maximal f-ascent. We need only consider the former two of these cases since for any f-ascent present in an edge ordering extended from these cases, its reverse will be present in one of the latter cases–with the roles of x_0 and y_0 switched with x_5 and y_5 respectively. These cases are shown in Figure 3 where the paths labelled $abcd$ and rst are f-ascents of S. Moving forward we will refer to the labels in this figure to simplify notation.

Firstly, suppose rst is a maximal f-ascent. Then $t > \pi$ and, since u_5 is not a 3-kernel of S, $\pi t\phi r$ is a $(4, f)$-ascent. But then $t < \phi < r < s < t$, which is a contradiction.

Secondly, suppose that $rst\pi$ is an f-ascent of S. If $r < b$, then since $t > r$, either rb (if $\phi > r$) or ϕrb (if $\phi < r$) is a maximal f-ascent, which in either case is a contradiction. Therefore we may assume $r > b$. We may also assume that $\phi > r$, or else abr is a u_5-avoiding maximal f-ascent. Furthermore, if $c > r$, then rcd is a maximal f-ascent, so we may assume $c < r$. Now if $\phi < s$, then ϕs is a u_5-avoiding maximal f-ascent, which is a contradiction. Thus we may assume $\phi > s$. Since $r < s$ by assumption, we now have $c < r < s < \phi$, which implies that $cs\phi$ is a maximal f-ascent, and again we have a contradiction.

This case completes the basis step of the proof.

Assume the result to be true for graphs in $\mathcal{S}$ constructed from K_2 in fewer than $k \geq 2$ steps. Consider any graph $S = S_k$ constructed from K_2 in k steps, and any edge ordering f of S.

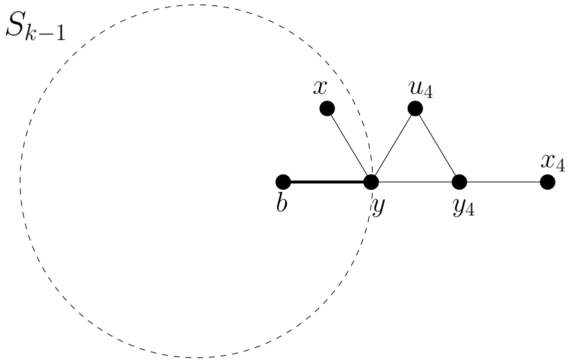

Figure 4: S is constructed by joining y to y_4 and u_4 of a new $P_3 : u_4, y_4, x_4$.

Suppose that in the construction of S one of **O1**, **O2** or **O5** was performed at least once. Then S contains $y \in Y_S$ such that y was joined to a new vertex in step $i \geq 2$ and such that y is incident with at least two bridges. Let $y \in Y_S$ be incident to at least two bridges, and x be the vertex of degree one adjacent to y. Note that one of the bridges incident with y is xy. Let $G_1, G_2, ..., G_m$ be the components of $S - y$ which consist of at least two vertices. For each $1 \leq i \leq m$, let G'_i be the subgraph induced by $\{x, y\} \cup V(G_i)$. Then each $G'_i \in \mathcal{S}_j$ for some $1 \leq j < k$. If $G'_i \cong S_j \in \mathcal{S}_j$, then let $U_{G'_i} = U_j$ and f'_i be the edge ordering of G'_i induced by f.

Since y is incident with a bridge other than xy, there exists an i, say $i = 1$, such that $\deg_{G'_1}(y) = 2$. Let $H = S - G_1$ and f_H be the edge ordering of H induced by f. Then $H \cong S_j \in \mathcal{S}_j$ for some $1 \leq j < k$. Let $U_H = U_j$. By the induction hypothesis there exists at least one U_H-avoiding maximal $(2, f_H)$- or $(3, f_H)$-ascent and we may assume that all such maximal f_H-ascents start or end at y, or else there exists a U_S-avoiding maximal f-ascent of length at most three in S and we are done. Without loss of generality assume that there exists a U_H-avoiding maximal f_H-ascent of length at most three which ends at y. Then by Lemma 8 there exists a maximal f_H-ascent $\lambda = aby$ or $\lambda = acby$ such that $f_H(by) = \max_{v \in N(y)}\{f_H(vy)\}$ and $a \in V(H) - U_H$.

Let b_1 be the neighbour of y in G_1. By the induction hypothesis, there exists at least one $U_{G'_1}$-avoiding maximal $(2, f'_1)$- or $(3, f'_1)$-ascent and we may assume that all such maximal f'_1-ascents start or end at y, or else we are done. Thus either b_1y is the initial or final edge of a $U_{G'_1}$-avoiding maximal f'_1-ascent α of length at most three. If α starts at y, then $f'_1(b_1y) < f(xy) < f(by)$ and λ is a U_S-avoiding maximal f-ascent of length at most three. If α ends at y, then in S either α (if $f'_1(b_1y) > f_H(by)$) or λ (if $f'_1(b_1y) < f_H(by)$) is a U_S-avoiding maximal f-ascent of length at most three.

Suppose then that only **O3** and **O4** are used in the construction of S.

Firstly, suppose that S is constructed from S_{k-1} by joining y to y_4 and u_4 of a new $P_3 : u_4, y_4, x_4$ (see Figure 4). Then $U_S = U_{k-1} \cup \{u_4\}$. Let f' be the edge ordering of S_{k-1} induced by f, and x the end vertex adjacent to y. By the induction hypothesis, in S_{k-1} there exists a U_{k-1}-avoiding maximal f'-ascent of length at most three. We may assume that all such f'-ascents start or end at y or else we are done. Without loss of generality assume that there exists a U_{k-1}-avoiding maximal f'-ascent of length at most three which ends at y. By Lemma 8 there exists a maximal f'-ascent $\lambda = aby$ or $\lambda = acby$ such that $f'(by) = \max_{v\in N(y)}\{f'(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$. If λ is a maximal f-ascent, then we are done so we may assume that either

$$f(yu_4) > f(by) \text{ or } f(yy_4) > f(by). \tag{1}$$

- Suppose $f(yu_4) > f(by)$. Then $f(yu_4) = \max_{v\in N(y)-y_4}\{f(vy)\}$.
  - If $f(y_4u_4) < f(u_4y)$, then either y_4u_4y or $x_4y_4u_4y$ is a U_S-avoiding maximal f-ascent.
  - Suppose $f(y_4u_4) > f(u_4y)$. Then $f(x_4y_4) > f(y_4u_4)$, or else $x_4y_4u_4$ is a U_S-avoiding maximal f-ascent.
    - If $f(yy_4) > f(y_4x_4)$, then $f(yy_4) = \max_{v\in N(y)}\{f(vy)\}$ and x_4y_4y is a U_S-avoiding a maximal f-ascent.
    - If $f(yy_4) < f(y_4x_4)$, then either xyy_4x_4 (if $f(xy) < f(yy_4)$) or y_4yx (if $f(xy) > f(yy_4)$) is a U_S-avoiding maximal f-ascent.
- Suppose then that $f(yu_4) < f(by)$. Then by (1), $f(yy_4) > f(by)$ and $f(yy_4) = \max_{v\in N(y)}\{f(vy)\}$. This implies either xyy_4x_4 (if $f(yy_4) < f(y_4x_4)$) or x_4y_4y (if $f(yy_4) > f(y_4x_4)$) is a maximal f-ascent, neither of which starts or ends in U_S.

Secondly, suppose that S is constructed from S_{k-1} by joining $y \in Y_{k-1}$ to the vertices v_3 and u_3 of a new edge u_3v_3. Then $U_S = U_{k-1} \cup \{u_3\}$. Let S' be the subgraph of S induced by $\{x, y, v_3, u_3\}$, f' the edge ordering of S' induced by f, and f'' the edge ordering of S_{k-1} induced by f. Note that $S' \cong S_1 \in \mathcal{S}_1$. Let $U_{S'} = \{u_3\}$. By the induction hypothesis, there exists a u_3-avoiding maximal f'-ascent α of length at most three. We may assume that α either starts or ends at y, or else we are done. Without loss of generality assume that α starts at y. Necessarily, $f'(yx) > f'(yu_3)$ and $\alpha = yu_3v_3$. Furthermore, we may assume that $f'(yv_3) > f'(yu_3)$, or else $f'(yv_3) < f(yu_3) < f(u_3v_3)$ and v_3yx is a U_S-avoiding maximal f-ascent of length two and we are done. Thus $f'(yu_3) = \min_{v\in N(y)}\{f'(vy)\}$.

By the induction hypothesis, there exists a U_{k-1}-avoiding maximal f''-ascent λ of length at most three in S_{k-1}. We may assume that λ starts or ends at y or else we are done. If λ starts at y, then by Lemma 8 there exists a maximal f''-ascent $\lambda' = aby$ or $\lambda' = acby$ such that $f''(by) = \min_{v\in N(y)}\{f''(vy)\}$ and $a \in V(S_{k-1}) - U_{k-1}$. This implies either λ' or α is a U_S-avoiding maximal f-ascent of length at most three. Assume then that λ ends at y, and furthermore, that all U_{k-1}-avoiding maximal f''-ascents of length at most three end at y. Then there exists an edge $vy \in E(S_{k-1})$ such that $f''(vy) < f'(yu_3)$ otherwise α is a U_S-avoiding maximal f-ascent of length two and we are done. Let wy be the edge in S_{k-1} such that $f''(wy) = \min_{v\in N(y)}\{f''(vy)\}$. Then $f''(wy) < f'(yu_3) < f'(yv_3)$ which implies $f(wy) = \min_{v\in N(y)}\{f(vy)\}$. Recall

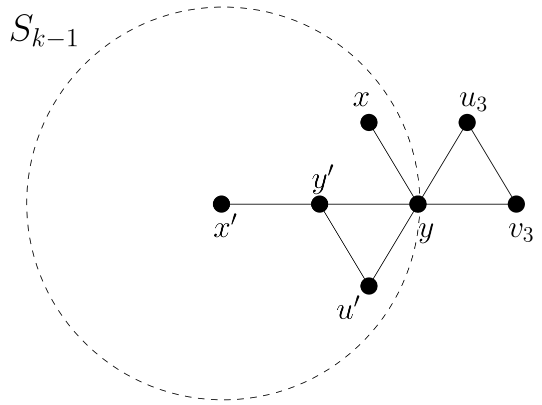

Figure 5: S is constructed from S_{k-1} by joining y to u_3 and v_3 of a new edge $\{u_3, v_3\}$.

that we have assumed S is constructed using only **O3** and **O4**, and that for any graph in $\mathcal{S}$, each vertex in $y \in Y_S$ is involved in **O3** at most once. Thus the edge wy was introduced by **O4**, which implies either $w = u' \in U_{k-1}$ and is adjacent to a vertex $y' \in Y_{k-1}$, or $w = y' \in Y_{k-1}$ and is adjacent to a vertex $u' \in U_{k-1}$. In either case, let x' be the vertex of degree one adjacent to y' – see Figure 5.

Suppose $w = y'$. If $f(x'y') < f(y'y)$, then, since $f(y'y) < f(xy)$, $x'y'yx$ is a U_S-avoiding maximal f-ascent of length three. If $f(x'y') > f(y'y)$, then, since $f(y'y) = \min_{v \in N(y)}\{f(vy)\}$, $yy'x'$ is a U_S-avoiding maximal f-ascent of length two.

Suppose then that $w = u'$. Let G_1 be the component of $S - y$ containing w, and G_1' the subgraph of S induced by $V(G_1) \cup \{y, x\}$. Then $G_1' \cong S_j \in \mathcal{S}_j$ for some $1 \leq j < k$. Let $U_{G_1'} = U_{S_j}$ and f_1' be the edge ordering of G_1' induced by f. By the induction hypothesis, there exists a $U_{G_1'}$-avoiding maximal f_1' ascent of length at most three in G_1'. Necessarily all $U_{G_1'}$-avoiding maximal f_1' ascent of length at most three start or end at y or else we are done. Suppose there exists such an ascent which starts at y. By Lemma 8 there exists a $U_{G_1'}$-avoiding maximal f_1' ascent λ of length at most three whose initial edge is $yw = yu'$. But since $f(yu') = \min_{v \in N(y)}\{f(yv)\}$, λ is also a U_S-avoiding maximal f-ascent which is a contradiction. Hence we may assume that there exists a $U_{G_1'}$-avoiding maximal f_1'-ascent λ of length at most three which ends at y. Since $f_1'(u'y) = \min_{v \in N(y)}\{f_1'(vy)\}$, $f_1'(u'y) > f_1'(xy)$ and the last edge of λ is $y'y$. This implies $f_1'(yy') > f_1'(xy)$ or equivalently, $f(y'y) > f(yx)$. Necessarily, $f(x'y') < f(y'y)$, or else $xyy'x'$ is a U_S-avoiding maximal f-ascent of length at most three. Now we look at three cases for the value of $f(y'u')$. In these cases we assume that $\deg_S(y') > 3$ or else either xyy' (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') > f(yy')$) is a U_S-avoiding maximal f-ascent.

Case 1 $f(yu') < f(y'u') < f(x'y')$. Then $yu'y'x'$ is a U_S-avoiding maximal f-ascent.

We define the following to aid us in the next two cases. Let H_1 be the component of $S_{k-1} - y'$ containing w, H_1' the the subgraph of S_{k-1} induced by $V(H_1) \cup \{y', x'\}$, and H_2' the subgraph of S_{k-1} induced by $V(S_{k-1}) - V(H_1)$. Then each $H_i \in \mathcal{S}_\ell$ for some $1 \leq \ell < k$. If $H_i' \cong S_\ell \in \mathcal{S}_\ell$, then let $U_{H_i'} = U_\ell$ and f_i be the edge ordering of

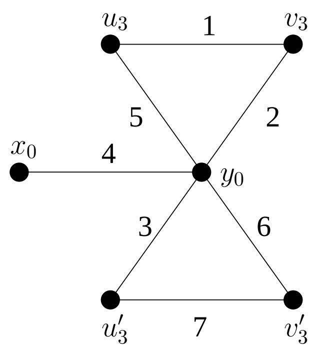

Figure 6: A graph G constructed from S_0 by performing **O3** twice at y_0, and an edge labelling f of G for which every maximal f-ascent of length at most three starts or ends in $U_G = \{u_3, u_3'\}$.

H_i' induced by f.

Case 2 $f(y'u') < f(x'y')$ and $f(y'u') < f(u'y)$. Then, in H_1', $y'u'yx$ is a $U_{H_1'}$-avoiding maximal f_1-ascent starting at y' and xyy' is a $U_{H_1'}$-avoiding maximal f_1-ascent ending at y. By the induction hypothesis, in H_2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three. We may assume that all such f_2-ascents start or end at y'. Without loss of generality suppose there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that ends at y'. By Lemma 8, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')}\{f_2(vy')\}$. Thus, in S, either λ or xyy' is a U_S-avoiding maximal f-ascent of length at most three.

Case 3 $f(y'u') > f(x'y')$. Then either xyy' (if $f(y'u') < f(yy')$) or $yu'y'$ (if $f(y'u') > f(yy')$) is a $U_{H_1'}$-avoiding maximal f_1-ascent which ends at y'. Again, by the induction hypothesis, in H_2, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three and we assume that all such f_2-ascents start or end at y'. Suppose there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that ends at y'. By Lemma 8, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \max_{v \in N(y')}\{f_2(vy')\}$. Therefore, in S, either λ, xyy', or $xu'y'$ is a U_S-avoiding maximal f-ascent of length at most three. Suppose then that there exists a $U_{H_2'}$-avoiding maximal f_2-ascent of length at most three that starts at y'. By Lemma 8, there exists a $U_{H_2'}$-avoiding maximal f_2-ascent $\lambda = aby'$ or $\lambda = acby'$ such that $f_2(by') = \min_{v \in N(y')}\{f_2(vy')\}$. Necessarily, $f(by') < f(y'x')$, and since $f(y'y) > f(y'x')$ and $f(y'u') > f(y'x')$, λ is a U_S-avoiding maximal f-ascent of length at most three. □

In the construction of $S_k \in \mathcal{S}_k$, any vertex $y \in Y_k$ is involved in **O3** at most once. If not, then U_k is no longer a 3-kernel of S_k. Consider the graph G shown in Figure 6, which is constructed from S_0 by performing **O3** twice at y_0. Let $U_G = \{u_3, u_3'\}$. For the edge labelling f of G shown in the figure, any maximal f-ascent of length at most three starts or ends in U_G.

Recall that the graphs $G_S \in \mathcal{G}$ are obtained from a graph $S \in \mathcal{S}$ by performing operations **O6** and **O7**. We now show that these graphs also have depression at most three.

Theorem 10. *For each $G_S \in \mathcal{G}$, $\varepsilon(G) \leq 3$.*

Proof. Let G'_S be constructed from $S \in \mathcal{S}$ by adding $n \geq 0$ edges between vertices in $Y_{G'_S} = Y_S$ and let $U_{G'_S} = U_S$. If $n = 0$, then $G'_S \in \mathcal{S}$ and by Theorem 9, $\varepsilon(G'(S)) \leq 3$ and $U_{G'_S}$ is a k-kernel of G'_S, where $k \in \{2, 3\}$.

Suppose that $n \geq 1$. Let f be an edge ordering of G'_S, and f' the edge ordering of S induced by f. If there exists a $(U_S \cup Y_S)$-avoiding maximal f'-ascent of length at most three, then $h(f) \leq 3$. Suppose then that there does not exist a $(U_S \cup Y_S)$-avoiding f'-ascent of length at most three. By Theorem 9 there exists a U_S-avoiding maximal f'-ascent of length at most three in S, thus all maximal U_S-avoiding $(2, f')$- or $(3, f')$-ascents start or end in Y_S.

Without loss of generality we assume there exists a maximal U_S-avoiding ascent of length at most three which ends in Y_S. By Lemma 8, S has property **P1**, which implies that there exists a maximal f'-ascent $\lambda = aby_1$ or $\lambda = acby_1$ such that $y_1 \in Y_S$ and $f'(by_1) = \max_{v \in N_S(y_1)}\{f'(vy_1)\}$. Suppose that in G'_S there exists an edge y_1w such that $f(y_1w) = \max_{v \in N_{G'_S}(y_1)}\{f(vy_1)\} > f(by_1)$ and w is not a vertex of λ. Necessarily, $y_1w \notin E(S)$ which implies $w \in Y_S$. Let $w = y_2$, and x_1 and x_2 be the vertices of degree one adjacent to y_1 and y_2 respectively. Since λ is a maximal f'-ascent in S, it follows that $f(y_1x_1) < f(by_1) < f(y_1y_2)$. Therefore, either $x_1y_1y_2x_2$ (if $f(y_2x_2) > f(y_1y_2)$) or $x_2y_2y_1$ (if $f(y_2x_2) < f(y_1y_2)$) is a $U_{G'_S}$-avoiding maximal f-ascent. Hence $U_{G'_S}$ is a k-kernel of G'_S, where $k \in \{2, 3\}$.

Let $G_S \in \mathcal{G}$ be constructed from G'_S by adding any set $A = A(G_S)$ of new vertices to G'_S and arbitrary edges between vertices in $A \cup U_{G'_S}$. Then by Theorem 3, $\varepsilon(G_S) \leq 3$. □

Note that $\kappa(G_S) = 1$ for each $G_S \in \mathcal{G}_S$. We also note that for each graph G in the classes of graphs with depression three defined in [6], [10], and [13], either $\text{diam}(L(G)) = 2$ or $\kappa(G) = 1$. The graph H shown in Figure 7 is an example of a graph with $\kappa(H) > 1$, $\text{diam}(L(H)) > 2$, and $\varepsilon(H) = 3$. We provide the following argument to support the claim that $\varepsilon(H) = 3$. Suppose to the contrary that $\varepsilon(H) > 3$. Let $f : E(H) \to \{1, 2, ..., 8\}$ be an edge ordering of H such that every maximal f-ascent has length at least 4. Since e_1 and e_8 are the only edges in H which are at distance three in $L(H)$, it follows that $\{f(e_1), f(e_8)\} = \{1, 8\}$. If not, then there exists a maximal f-ascent of length at most three which begins and ends with the edges assigned 1 and 8 under f, a contradiction.

Without loss of generality we may assume that $f(e_1) = 1$ and $f(e_8) = 8$. Without loss of generality we may also assume that $f(e_5) = \max\{f(e_2), f(e_3), f(e_4), f(e_5)\}$. Then, since $h(f) > 3$ and $f(e_4) < f(e_5)$, it follows that $e_7e_2e_4e_5$ is a maximal f-ascent. However, this implies e_1e_2 is a maximal f-ascent, a contradiction.

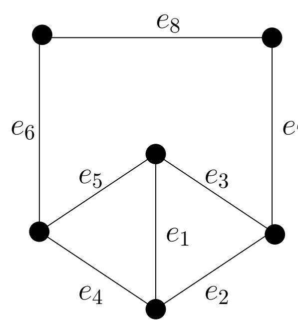

Figure 7: A graph H with $\kappa(H) > 1$, $\text{diam}(L(H)) > 2$, and $\varepsilon(H) = 3$.

5 Open Problems

1. Characterize the class of graphs with depression three.
2. Does there exist a finite number of operations of the type **O1-O7** that would yield all graphs with depression three?
3. Use a similar construction to produce large classes of graphs with depression $k \geq 4$.

References

[1] A. Bialostocki and Y. Roditty, A monotone path in an edge-ordered graph, *Internat. J. Math. Math. Sci.* 10(2) (1987), 315–320.

[2] A. P. Burger, E. J. Cockayne and C. M. Mynhardt, Altitude of small complete and complete bipartite graphs, *Australas. J. Combin.* 31 (2005), 167–177.

[3] A. R. Calderbank, F. R. K. Chung and D. G. Sturtevant, Increasing sequences with nonzero block sums and increasing paths in edge-ordered graphs, *Discrete Math.* 50(1) (1984), 15–28.

[4] G. Chartrand and L. M. Lesniak, *Graphs and digraphs* (3rd ed.), Wadsworth & Brooks / Cole mathematics series, Wadsworth, 1996.

[5] V. Chvátal and J. Komlós, Some combinatorial theorems on monotonicity, *Canad. Math. Bull.* 14 (1971), 151–157.

[6] E. J. Cockayne, G. Geldenhuys, P. J. P. Grobler, C. M. Mynhardt and J. H. van Vuuren, The depression of a graph, *Util. Math.* 69 (2006), 143–160.

[7] I. Gaber-Rosenblum and Y. Roditty, The depression of a graph and the diameter of its line graph, *Discrete Math.* 309(6) (2009), 1774–1778.

[8] R. L. Graham and D. J. Kleitman, Increasing paths in edge ordered graphs, *Period. Math. Hungar.* 3 (1973), 141–148. Collection of articles dedicated to the memory of Alfréd Rényi, II.

[9] J. Katrenič and G. Semanišin, Finding monotone paths in edge-ordered graphs, *Discrete Appl. Math.* 158(15) (2010), 1624–1632.

[10] C. M. Mynhardt, Trees with depression three, *Discrete Math.* 308(5-6) (2008), 855–864.

[11] C. M. Mynhardt, A. P. Burger, T. C. Clark, B. Falvai and N. D. R. Henderson, Altitude of regular graphs with girth at least five, *Discrete Math.* 294(3) (2005), 241–257.

[12] C. M. Mynhardt and M. Schurch, The depression of a graph and k-kernels, *Discussiones Math.* (to appear).

[13] C. M. Mynhardt and M. Schurch, A class of graphs with depression three, *Discrete Math.* 313 (2013), 1224–1232.

[14] Y. Roditty, B. Shoham and R. Yuster, Monotone paths in edge-ordered sparse graphs, *Discrete Math.* 226(1-3) (2001), 411–417.

[15] M. Schurch, *On the Depression of Graphs*, Doctoral dissertation, University of Victoria, April 2013. `http://hdl.handle.net/1828/4527`

[16] R. Yuster, Large monotone paths in graphs with bounded degree, *Graphs Combin.* 17(3) (2001), 579–587.

(Received 1 Jan 2013; revised 3 Nov 2013)

Among the class of graphs with a bounded cop number one can mention the class of trees, which can easily shown to have cop number one, and the class of planar graphs, which have cop number at most three: Theorem 1 ([1]).Â Note that either result can be used to construct geometric graphs of cop number three. The construction provided in the proof of Theorem A is based on obtaining a polygonal-curve embedding from a given straight-line embedding of G and then subdividing the edge-curve equally such that with an appropriate parameter for the geometric graphs, the resulting embedding is geometric. The latter, for example, requires that no subdividing vertex on an edge-polygonal curve be adjacent to a vertex belonging to another edge-polygonal curve. A decision tree is a decision support tool that uses a tree-like graph or model of decisions and their possible consequences, including chance event outcomes, resource costs, and utility. It is one way to display an algorithm that only contains conditional control statements. A decision tree is a flowchart-like structure in which each internal node represents a â€œtestâ€ on an attribute (e.g. whether a coin flip comes up heads or tails), each branch represents the outcome of the test, and each leaf node represents a class label (decision taken after computing all attributes).Â â€œThe possible solutions to a given problem emerge as the leaves of a tree, each node representing a point of deliberation and decision.â€ - Niklaus Wirth (1934 â€”), Programming language designer.